

A NEW q -ANALOGUE FOR BERNOULLI NUMBERS

O-YEAT CHAN AND DANTE MANNA

ABSTRACT. Inspired by [2], we define a new sequence of q -analogues for the Bernoulli numbers under the framework of Strodtt operators. We show that they not only satisfy many identities similar to those of the q -analogue proposed by Carlitz [3], but also interesting analytical properties as functions of q . In particular, we give a simple analytic proof of a generalization of an explicit formula for the Bernoulli numbers given by Woon [15]. We also define a set of q -analogues for the Stirling numbers of the second kind within our framework and prove a q -extension of a related, well-known closed form relating Bernoulli and Stirling numbers.

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1. INTRODUCTION

The Bernoulli numbers $\{B_n\}_{n \geq 0}$ are rational numbers in a sequence defined by the binomial recursion formula

$$(1.1) \quad \sum_{k=0}^m \binom{m}{k} B_k - B_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1, \end{cases}$$

or equivalently, the generating function

$$(1.2) \quad \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{t}{e^t - 1}.$$

This sequence appears in a multitude of diverse contexts, including number theory, combinatorics, and finite approximations of integrals. A short list of applications can be found at [8].

q -analogues of the Bernoulli numbers were first studied by Carlitz [3, 4] in the middle of the last century when he introduced a power of q to (1.1), giving a new sequence $\{\beta_m, m \geq 0\}$:

$$(1.3) \quad \sum_{k=0}^m \binom{m}{k} \beta_k q^{k+1} - \beta_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases}$$

Here, and in the remainder of the paper, we make the assumption that $|q| < 1$. Clearly we recover (1.1) if we let $q \rightarrow 1$ in (1.3).

Since Carlitz, there have been many distinct q -analogues of Bernoulli numbers arising from varying motivations. Tsumura's [14] definition comes from a generalization of a Dirichlet's series, while the q -Bernoulli numbers due to Suslov [13] come from a generalized Fourier series. Improvements to the latter work were proposed by Ismail and Rahman in [12], who produced a bivariate extension of Bernoulli polynomials that comes from the inverse of an Askey-Wilson operator. Because of their relation to Fourier series, these are also associated with a q -generalization of the Riemann zeta function.

In this paper we propose a q -analogue of Bernoulli numbers which is different from those listed above. Our definition is motivated by a development of Bernoulli numbers that appears in [2], which unifies the definition of Bernoulli and Euler numbers under the theory of Strodtt Operators. The resulting sequence $\{\beta_n\}_{n \geq 0}$ has natural q -analogues of identities (1.1) and (1.2), which we construct

in Section 3, along with a closed form in Section 4. We go on to prove a number of analytic properties of the $\tilde{\beta}_n$ as rational functions of q , including information about its zeroes and poles. Next, we define q -Bernoulli polynomials using a q -analogue of x^n in Section 5, and use this to generalize the classical identity relating Bernoulli polynomials to power sums. This culminates in another closed form which relates our q -Bernoulli numbers to a q -analogue of Stirling numbers of the second kind, which we discuss in Section 6.

Throughout our work we will compare the properties of our sequence $\tilde{\beta}_n$ to those of Carlitz sequence β_n . Because of the way it is defined, our sequence is primarily combinatorial in nature, and so, of the generalizations listed above, it is most comparable to the β_n . This not only provides a point of reference to the suitability of $\tilde{\beta}_n$ as a “good” q -analogue, but also highlights the differences between it and its closest relative among the myriad of q -analogues in existence.

Before we continue, let us introduce some notation that is used in the remainder of the paper. The q -analogue of an integer is denoted using the q -bracket notation

$$[x]_q := \frac{q^x - 1}{q - 1},$$

for all $x \in \mathbb{R}$ and $q > 0$. We also define the q -rising factorial or the q -Pochhammer symbol in the usual way:

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

with

$$(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n$$

and $(a; q)_0 := 1$. We will also make significant use of the well-known q -binomial theorem [1, p. 17, Thm. 2.1]:

Theorem 1.1. *For $|q|, |z| < 1$ and any complex a we have*

$$(1.4) \quad \frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n,$$

and the limiting case

$$(1.5) \quad (a; q)_\infty = \sum_{n \geq 0} \frac{(-1)^n a^n q^{n(n-1)/2}}{(q; q)_n}.$$

We also define the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

In general, a q -binomial coefficient is a polynomial in q (also called a *Gaussian polynomial*) whose q^j -coefficient counts the number of partitions of j with at most k parts each less than or equal to $n - k$ [1, p. 33].

Finally, we introduce the following convenient shorthand analogue to the factorial:

$$[k]_q! := [k]_q [k-1]_q \cdots [1]_q = \frac{(q; q)_k}{(1-q)^k}.$$

2. BASIC PROPERTIES OF THE CARLITZ q -ANALOGUE

We begin by collecting some basic facts about the Carlitz sequence β_n . It is obvious from (1.3) that the β_n will always be rational in q , and we easily compute the first few values:

$$\begin{aligned}\beta_0 &= 1, \\ \beta_1 &= -\frac{1}{[2]_q}, \\ \beta_2 &= \frac{q}{[3]_q[2]_q}, \\ \beta_3 &= -\frac{q(q-1)}{[4]_q[3]_q}, \\ \beta_4 &= \frac{q(q^4 - q^3 - 2q^2 - q + 1)}{[5]_q[4]_q[3]_q}, \\ \beta_5 &= -\frac{q(q-1)(q^4 - 2q^3 - 3q^2 - 2q + 1)}{[6]_q[5]_q[4]_q}.\end{aligned}$$

An analogue of (1.2) also exists for the β_n . Define

$$B(t) := \sum_{m=0}^{\infty} \beta_m \frac{t^m}{m!}.$$

Multiplying both sides of (1.3) by $\frac{t^m}{m!}$ and summing over m , we get

$$\beta_0 + \beta_1 t + \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{m}{k} \beta_q q^{k+1} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta_m \frac{t^m}{m!} + \beta_0 q + \beta_0 q t + \beta_1 q^2 t$$

We observe that the double sum is a product of single sums and use this to rewrite this formula as

$$(2.1) \quad B(t) = 1 - q - t + qe^t B(qt).$$

Iterating, we obtain

$$\begin{aligned}(2.2) \quad B(t) &= (1 - q) \sum_{m=0}^{\infty} q^m e^{[m]_q t} - t \sum_{m=0}^{\infty} q^{2m} e^{[m]_q t} \\ &= \sum_{m=0}^{\infty} (1 - q - tq^m) q^m e^{[m]_q t},\end{aligned}$$

which converges as long as $|q| < 1$. It is clear that letting $q \rightarrow 1$ recovers (1.2).

3. NEW q -BERNOULLI NUMBERS

In [2], the authors define the *Strodt numbers* corresponding to a certain probability distribution $\{P_n\}_{n \geq 0}$ via the exponential generating function

$$\sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = \frac{1}{Q(t)},$$

where $Q(t)$ is the moment generating function of the distribution. They showed that the Strodt numbers of the uniform distribution on $[0, 1]$, are actually Bernoulli numbers, while the average of two Dirac delta functions at 0 and 1 yield the Euler numbers.

We define our q -analogue using this framework, replacing the integral on $[0, 1]$ with its corresponding q -integral. The Jackson q -integral of a function f is defined by

$$\int_0^a f(x) d_q x := \sum_{n=0}^{\infty} f(aq^n)(aq^n - aq^{n+1}).$$

Using this definition, the n th q -moment of the uniform distribution on $[0, 1]$ is

$$\tilde{\mu}_n := \int_0^1 x^n d_q x = \frac{1}{[n+1]_q}.$$

Continuing with the moments $\tilde{\mu}_n$, we use a q -exponential generating function (with $|t| < 1$) to obtain

$$\sum_{n=0}^{\infty} \tilde{\mu}_n \frac{t^n}{(q; q)_n} = (1-q) \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_{n+1}} = \frac{1-q}{t} \left(\frac{1}{(t; q)_{\infty}} - 1 \right).$$

The second equality follows from (1.4), with $z = t$ and $a = 0$.

As in [2], we now obtain the generating function for our q -analogue of the Bernoulli numbers $\tilde{\beta}_n$ by inverting the moment generating function:

$$(3.1) \quad F(t, q) := \sum_{n=0}^{\infty} \tilde{\beta}_n \frac{t^n}{(q; q)_n} := \frac{t}{1-q} \left(\frac{1}{\frac{1}{(t; q)_{\infty}} - 1} \right).$$

Upon replacing t by $t(1-q)$, and using (1.4) again, it is easy to see that we regain (1.2) as q tends to 1.

It is also simple to see from here that an analogue of (1.1) holds.

Proposition 3.1. *The q -Bernoulli numbers $\tilde{\beta}_n$ defined by (3.1) satisfy the q -binomial recurrence*

$$(3.2) \quad \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q \tilde{\beta}_k - \tilde{\beta}_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1. \end{cases}$$

Proof. We see that

$$F(t, q) \frac{1}{(t; q)_{\infty}} - F(t, q) = \frac{t}{1-q},$$

and by multiplying power series on the left using (1.4), and matching t^m -coefficients, the statement follows. \square

It is in (3.2) where a major difference to Carlitz's q -analogue is seen. Instead of a power of q appearing in the binomial recursion, the binomial itself is replaced by its q -analogue. We use this formula to calculate the first few $\tilde{\beta}_n$, confirming that this sequence is in fact substantially different from the β_n .

$$\begin{aligned} \tilde{\beta}_0 &= 1, \\ \tilde{\beta}_1 &= -\frac{1}{[2]_q}, \\ \tilde{\beta}_2 &= \frac{q^2}{[2]_q [3]_q}, \\ \tilde{\beta}_3 &= -\frac{q^3(q-1)}{[2]_q [4]_q}, \\ \tilde{\beta}_4 &= \frac{q^4(q^6 - q^4 - 2q^3 - q^2 + 1)}{[2]_q^2 [3]_q [5]_q} \end{aligned}$$

$$\tilde{\beta}_5 = -\frac{q^5(q-1)(q^6 - q^5 - q^4 - q^3 - q^2 - q + 1)}{[2]_q^2[6]_q}.$$

4. PROPERTIES AND CLOSED FORMULAS

In this section we collect some of the properties of the $\tilde{\beta}_n$ and derive a closed-form formula in terms of q -multinomial coefficients, which gives us a new proof of the corresponding combinatorial closed form for B_n . We begin by proving a structural property.

Proposition 4.1. *For all $n \geq 0$, the scaled q -Bernoulli numbers $[n+1]_q \tilde{\beta}_n$ are integer linear combinations of rational functions of q of the form*

$$\frac{[n+1]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q! [n+1-\kappa]_q!},$$

where $m \geq 0$ and the k_i are positive integers such that

$$\kappa := k_1 + k_2 + \cdots + k_m - m \leq n + 1.$$

We briefly remark that the form of the terms in the previous proposition in q -Pochhammer notation is

$$\frac{(q^{n+2+k_1+\cdots+k_m-m}; q)_{k_1+\cdots+k_m-m} (1-q)^m}{(q; q)_{k_1} \cdots (q; q)_{k_m}}$$

and the term is equal to 1 when $m = 0$.

Proof. We use strong induction on n . The claim is satisfied in the base case $\tilde{\beta}_0 = 1$ using $m = 0$.

Now assume that the claim holds for all $0 \leq k \leq n-1$. From (3.2) we can see that

$$[n+1]_q \tilde{\beta}_n = - \sum_{k=0}^{n-1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \tilde{\beta}_k.$$

An arbitrary rational term from an arbitrary $\tilde{\beta}_k$, where $0 \leq k \leq n-1$, by assumption, can be written as an integer times

$$\frac{1}{[k+1]_q} \cdot \frac{[k+1]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q! [k+1-\kappa]_q!}.$$

This term is multiplied by $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q$, leaving an integer times

$$\frac{[n+1]_q!}{[k]_q! [n-k+1]_q!} \cdot \frac{[k]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q! [k+1-\kappa]_q!} = \frac{[n+1]_q!}{[n-k+1]_q! [k_1]_q! [k_2]_q! \cdots [k_m]_q! [n+1-\kappa']_q!},$$

with $\kappa' = (n-k+1) + k_1 + \cdots + k_m - (m+1) = \kappa - (k+1) + (n+1) \leq n+1$ by the inductive hypothesis. Therefore, the structure of the term in the integer linear expansion of $\tilde{\beta}_n$ is as claimed. \square

Example 4.1. As illustrations of the previous proposition, we offer

$$[4]_q \tilde{\beta}_3 = -1 + 2 \frac{[4]_q!}{[2]_q! [3]_q!} - \frac{[4]_q!}{([2]_q!)^2 [2]_q!}$$

and

$$[6]_q \tilde{\beta}_5 = -1 + 2 \frac{[6]_q!}{[2]_q! [5]_q!} - 3 \frac{[6]_q!}{([2]_q!)^2 [4]_q!} + 4 \frac{[6]_q!}{([2]_q!)^3 [3]_q!} - \frac{[6]_q!}{([2]_q!)^4 [2]_q!} + 2 \frac{[6]_q!}{[3]_q! [4]_q!} - 3 \frac{[6]_q!}{[2]_q! [3]_q! [3]_q!}.$$

The sign and value of the integer coefficient to which each term is multiplied have combinatorial interpretations which are explained in Theorem 4.2.

4.1. **Closed form for $\tilde{\beta}_n$.** We now derive a closed form for the q -Bernoulli numbers $\tilde{\beta}_n$ in terms of q -multinomial coefficients summed over partitions of n . Our theorem will imply the result of Proposition 4.1, but first we need a few definitions.

Define the q -multinomial for positive integers n , and nonnegative integers k_1, k_2, \dots, k_m , where $m \geq 2$ and $\sum_i k_i = n$, by the formula

$$\begin{bmatrix} n \\ k_1, k_2, \dots, k_m \end{bmatrix}_q = \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q!} = \begin{bmatrix} n \\ k_1 \end{bmatrix}_q \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k_{m-1} \\ k_m \end{bmatrix}_q.$$

Note that:

- As q tends to 1 and we get $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k_1, k_2, \dots, k_m \end{bmatrix}_q = \binom{n}{k_1, k_2, \dots, k_m}$, the classical multinomial coefficient.
- The q -multinomials also generalize q -binomial coefficients, since $\begin{bmatrix} n \\ k, n-k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

We also have the following result due to P. A. MacMahon [1, p. 41, Thm. 3.6]:

Theorem 4.1. *In a non-commutative zero character ring with indeterminates $\{x_i\}_{i \geq 0}$, and q , such that $x_j x_i = q x_i x_j$ for all $i < j$, and $q x_i = x_i q$ for all i ,*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{i_1 + i_2 + \cdots + i_k = n} \begin{bmatrix} n \\ i_1, i_2, \dots, i_k \end{bmatrix}_q x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

for all integers $k \geq 1$, $i \geq 0$.

That is, the coefficient of q^m in $\begin{bmatrix} n \\ i_1, i_2, \dots, i_k \end{bmatrix}_q$ counts the number of permutations $\xi_1 \cdots \xi_n$ of the n -element multiset $\{1^{i_1}, \dots, k^{i_k}\}$ (where the multiplicity of j in the multiset is i_j , with $i_1 + \cdots + i_k = n$) such that exactly m pairs of the ξ_i, ξ_j appear with $i < j$ but $\xi_i > \xi_j$.

Let $\mathcal{C}(n)$ denote the set of integer compositions (ordered partitions) of n . Hence an element of $\mathcal{C}(n)$ is an ordered m -tuple of integers $p = (k_1, k_2, \dots, k_m)$, for some $1 \leq m \leq n$, where each $k_i > 0$ and $\sum_i k_i = n$. We also define the length of the composition as the length of the tuple: $\ell(p) = m$.

We define almost similarly the elements of $\mathcal{P}(n)$, the set of (unordered) partitions of n , except that the integers in each tuple are arranged in non-increasing order so that all partitions with the same parts are equivalent. Thus the set $\mathcal{C}(n)$ is bigger than, and contains, the set $\mathcal{P}(n)$. Let $NC(p)$ be the number of compositions that correspond to the partition p . By conventional counting techniques, it is clear that

$$NC(p) = \frac{\ell(p)!}{j_1! j_2! \cdots j_n!},$$

where j_i is the number of i s in the partition p , for all $1 \leq i \leq n$.

Example 4.2. Here is an enumeration of $\mathcal{C}(n)$ and $\mathcal{P}(n)$, including lengths and number of compositions for each partition, for $n = 4$.

$\mathcal{C}(4)$		$\mathcal{P}(4)$		
p	$\ell(p)$	p	$\ell(p)$	$NC(p)$
(4)	1	(4)	1	1
(3, 1)	2	(3, 1)	2	2
(1, 3)	2	(2, 2)	2	1
(2, 2)	2	(2, 1, 1)	3	3
(2, 1, 1)	3	(1, 2, 1)	3	3
(1, 2, 1)	3	(1, 1, 2)	3	3
(1, 1, 2)	3	(1, 1, 1, 1)	4	1
(1, 1, 1, 1)	4			

We are now ready to state the main theorem of this section.

Theorem 4.2. *For all $n \geq 1$,*

$$(4.1) \quad \tilde{\beta}_n = \frac{[n]_q!}{[2n]_q!} \sum_{p \in \mathcal{C}(n)} (-1)^{\ell(p)} \left[\begin{matrix} 2n \\ k_1 + 1, k_2 + 1, \dots, k_{\ell(p)} + 1, 1, \dots, 1 \end{matrix} \right]_q$$

$$(4.2) \quad = \frac{[n]_q!}{[2n]_q!} \sum_{p \in \mathcal{P}(n)} (-1)^{\ell(p)} NC(p) \left[\begin{matrix} 2n \\ k_1 + 1, k_2 + 1, \dots, k_{\ell(p)} + 1, 1, \dots, 1 \end{matrix} \right]_q,$$

where in the sum we have $(k_1, k_2, \dots, k_{\ell(p)}) = p$ in our multinomial, followed by $(n - \ell(p))$ 1s.

Proof. We prove the first of the two statements by strong induction on n . (The second follows from the first by reindexing.) Begin by noticing that

$$\tilde{\beta}_1 = \frac{-1}{[2]_q} = \frac{[1]_q!}{[2]_q!} (-1)^1 \left[\begin{matrix} 2 \\ 2 \end{matrix} \right]_q,$$

which proves the statement for $n = 1$ since (1) is the only integer composition of 1.

Now assume that the statement is true for all $1 \leq n < k$ where $k > 1$. By the recursive definition of $\tilde{\beta}_n$,

$$\begin{aligned} \tilde{\beta}_k &= - \sum_{i=0}^{k-1} \left[\begin{matrix} k \\ i \end{matrix} \right]_q \frac{\tilde{\beta}_i}{[k-i+1]_q} = - \frac{\tilde{\beta}_0}{[k+1]_q} - \sum_{i=1}^{k-1} \frac{[k]_q!}{[i]_q! [k-i]_q! [k-i+1]_q} \tilde{\beta}_i \\ &= - \frac{1}{[k+1]_q} + \frac{[k]_q!}{[2k]_q!} \sum_{i=1}^{k-1} \frac{(-1)[2k]_q!}{[i]_q! [k-i+1]_q!} \cdot \frac{[i]_q!}{[2i]_q!} \sum_{p \in \mathcal{C}(i)} (-1)^{\ell(p)} \left[\begin{matrix} 2i \\ i_1 + 1, i_2 + 1, \dots, i_{\ell(p)} + 1, 1, \dots, 1 \end{matrix} \right]_q, \end{aligned}$$

where the multinomial is $(i_1, i_2, \dots, i_{\ell(p)}) = p$, followed by $(i - \ell(p))$ 1s, by the induction hypothesis. Now rearrange factors to produce

$$\tilde{\beta}_k = - \frac{1}{[k+1]_q} + \frac{[k]_q!}{[2k]_q!} \sum_{i=1}^{k-1} \sum_{p \in \mathcal{C}(i)} -(-1)^{\ell(p)} \left[\begin{matrix} 2k \\ k-i+1, i_1+1, i_2+1, \dots, i_{\ell(p)}+1, 1, \dots, 1 \end{matrix} \right]_q.$$

The q -multinomial coefficient within the sum is indeed well-defined since our hypothesis implies

$$(i_1 + 1) + (i_2 + 1) + \dots + (i_{\ell(p)} + 1) + i - \ell(p) = 2i,$$

hence

$$(k - i + 1) + (i_1 + 1) + (i_2 + 1) + \dots + (i_{\ell(p)} + 1) + k - \ell(p) = 2k.$$

We rewrite the term that precedes the summation to obtain

$$- \frac{1}{[k+1]_q} = \frac{[k]_q!}{[2k]_q!} (-1)^1 \frac{[2k]_q!}{[k+1]_q!} = \frac{[k]_q!}{[2k]_q!} (-1)^1 \left[\begin{matrix} 2k \\ k+1, 1, \dots, 1 \end{matrix} \right]_q,$$

where the string of ones in the multinomial is of length $k - 1$. Hence this term corresponds to the exceptional singleton composition (k) .

All other compositions of k have multiple terms, and thus can be written uniquely in the form $(k - i, i_1, i_2, \dots, i_m)$, where (i_1, i_2, \dots, i_m) is a composition of i and $1 \leq i \leq k - 1$. We see that the length of the composition of k is one more than the length of the corresponding composition of i . The statement for k thus follows by reindexing the double sum as a single sum over compositions of k . □

Note 4.3. In Example 4.1 we noted the existence of certain integer coefficients multiplying the rational functions. The second of these two formulas clearly shows that the values of these coefficients are the $NC(p)$ —they count the number of ordered compositions for each partition; and the signs of these coefficients are $(-1)^{\ell(p)}$ —they express the parity of the length of each partition.

The previous theorem has the following immediate consequence, from taking the limit as $q \rightarrow 1$.

Corollary 4.1. *For all $n \geq 1$,*

$$(4.3) \quad B_n = \frac{n!}{(2n)!} \sum_{p \in \mathcal{C}(n)} (-1)^{\ell(p)} \binom{2n}{k_1 + 1, k_2 + 1, \dots, k_{\ell(p)} + 1, 1, \dots, 1}$$

$$(4.4) \quad = \frac{n!}{(2n)!} \sum_{p \in \mathcal{P}(n)} (-1)^{\ell(p)} NC(p) \binom{2n}{k_1 + 1, k_2 + 1, \dots, k_{\ell(p)} + 1, 1, \dots, 1},$$

where in the sum we have $(k_1, k_2, \dots, k_{\ell(p)}) = p$ in our multinomial, followed by $(n - \ell(p))$ 1s.

Remark 4.3. This kind of closed form for the Bernoulli numbers has been described in detail by Woon [15], who carefully constructs a tree and then sums its nodes to get a similar expression. However, no such result for the β_n appears in the literature.

4.2. Properties of $F(t, q)$. At this point we turn to the properties of the generating function $F(t, q)$, starting with a recursion. From the representation (3.1), we have that

$$\frac{F(t, q)}{(t; q)_\infty} - F(t, q) = \frac{t}{1 - q}.$$

After replacing t by tq , we have

$$F(tq, q) \frac{1 - t - (t; q)_\infty}{1 - (t; q)_\infty} = \frac{tq}{1 - q} \cdot \frac{(tq; q)_\infty (1 - t - (t; q)_\infty)}{(1 - (tq; q)_\infty)(1 - (t; q)_\infty)} = qF(t, q).$$

To obtain the last equality we use $(1 - t)(tq; q)_\infty = (t; q)_\infty$. But

$$\frac{1 - t - (t; q)_\infty}{1 - (t; q)_\infty} = 1 - \frac{t}{1 - (t; q)_\infty} = 1 - ((1 - q)F(t; q) + t).$$

Therefore,

$$(4.5) \quad \frac{1 - t}{1 - q} F(tq, q) - \frac{q}{1 - q} F(t, q) = F(tq, q)F(t, q).$$

Equating coefficients of t^n we get the following identity.

Proposition 4.2. *For all $n \geq 1$,*

$$(4.6) \quad \frac{1}{1 - q} \left(\frac{\tilde{\beta}_n q^n}{(q; q)_n} - \frac{\tilde{\beta}_{n-1} q^{n-1}}{(q; q)_{n-1}} \right) - \frac{q \tilde{\beta}_n}{(1 - q)(q; q)_n} = \sum_{k=0}^n \frac{\tilde{\beta}_k q^k \tilde{\beta}_{n-k}}{(q; q)_k (q; q)_{n-k}}.$$

Proposition 4.2 is a q -analogue of the quadratic recurrence [7, Eq. 24.14.2]

$$\sum_{k=0}^n \binom{n}{k} B_k B_{n-k} = (1 - n)B_n - nB_{n-1}.$$

It is worth noting that the recursion (4.5) does *not* lead to a continued fraction expansion for F . Indeed, isolating $F(t, q)$ gives

$$F(t, q) = \frac{\frac{1 - t}{1 - q} F(tq, q)}{F(tq, q) + \frac{q}{1 - q}} = \frac{1 - t}{1 - q + \frac{q}{F(tq, q)}},$$

which, when iterated, recovers the definition of F .

We may also derive a differential equation for $F(t, q)$. If we differentiate both sides of (3.1) with respect to t , we find that

$$\frac{\partial}{\partial t} F(t, q) = \frac{F(t, q)}{t} - \frac{t}{1 - q} \cdot \frac{(t; q)_\infty}{(1 - (t; q)_\infty)^2} \sum_{n \geq 0} \frac{q^n}{1 - tq^n}$$

$$\begin{aligned} &= \frac{F(t, q)}{t} \left(1 - \frac{(1-q)F(t, q)}{(t; q)_\infty} \sum_{n, k \geq 0} q^{n+kn} t^k \right) \\ &= \frac{F(t, q)}{t} \left(1 - \frac{(1-q)F(t, q)}{(t; q)_\infty} \sum_{k \geq 0} \frac{t^k}{1-q^{k+1}} \right). \end{aligned}$$

Noting that

$$\frac{F(t, q)}{(t; q)_\infty} = F(t, q) - \frac{t}{1-q},$$

we have

$$\frac{\partial}{\partial t} F(t, q) = F(t, q) \left(\frac{1-F(t, q)}{t} + \frac{1}{1-q} + (t - (1-q)F(t, q)) \sum_{k \geq 1} \frac{t^{k-1}}{1-q^{k+1}} \right).$$

If we differentiate with respect to q , we instead obtain

$$\frac{\partial}{\partial q} F(t, q) = \frac{1}{1-q} F(t, q) - \frac{t}{1-q} \frac{(t; q)_\infty}{(1 - (t; q)_\infty)^2} \sum_{n \geq 0} \frac{ntq^{n-1}}{1-tq^n}.$$

Combining this with the t derivative we get the partial differential equation

Proposition 4.3.

$$(4.7) \quad \frac{\partial F}{\partial t} \sum_{n \geq 0} \frac{ntq^{n-1}}{1-tq^n} - \frac{\partial F}{\partial q} \sum_{n \geq 0} \frac{q^n}{1-tq^n} = F(t, q) \left(\sum_{n \geq 0} \frac{nq^{n-1}}{1-tq^n} - \frac{1}{1-q} \sum_{n \geq 0} \frac{q^n}{1-tq^n} \right).$$

4.3. Analytic Properties of $\tilde{\beta}_n$ and its Zeroes and Poles. We now turn our attention to the study of $\tilde{\beta}_n$ as rational functions of q . The *degree* of a rational function $r(x) = \frac{a(x)}{b(x)}$ is commonly defined as the maximum of the polynomial degrees of numerator and denominator; that is, $\deg(r) := \max\{\deg(a), \deg(b)\}$. On the other hand, we define the *total degree* $\text{Tdeg}(r)$ as the difference in degrees: $\text{Tdeg}(r) = \deg(a) - \deg(b)$. The total degree of a rational function gives an asymptotic as $x \rightarrow \infty$, since $r(x) \sim x^{\text{Tdeg}(r)}$.

Proposition 4.4. (Properties of $\text{Tdeg}(r)$.) For any rational functions r_1, r_2 ,

- (a) $\text{Tdeg}(r_1 r_2) = \text{Tdeg}(r_1) + \text{Tdeg}(r_2)$,
- (b) $\text{Tdeg}(r_1 + r_2) = \max\{\text{Tdeg}(r_1), \text{Tdeg}(r_2)\}$, provided that $\text{Tdeg}(r_1) \neq \text{Tdeg}(r_2)$,
- (c) For all $0 \leq k \leq n$, $\text{Tdeg}(\binom{n}{k}_q) = k(n-k)$ (as rational functions in the variable q).

Proof. (a) and (b) are straightforward.

(c). Using (a), we see that

$$\begin{aligned} \text{Tdeg}(\binom{n}{k}_q) &= \text{Tdeg}([n]_q!) - \text{Tdeg}([k]_q!) - \text{Tdeg}([n-k]_q!) \\ &= \frac{(n+1)n}{2} - \frac{(k+1)k}{2} - \frac{(n-k+1)(n-k)}{2} = k(n-k). \end{aligned}$$

□

Proposition 4.5. For all $n \geq 0$,

$$\text{Tdeg}(\tilde{\beta}_n) = \frac{n(n-3)}{2}$$

as a rational function in the variable q .

Proof. We use strong induction on n .

Since $\tilde{\beta}_0 = 1$, $\text{Tdeg}(\tilde{\beta}_0) = 0$, so the base case holds.

For the inductive step, suppose that $\text{Tdeg}(\tilde{\beta}_k) = \frac{k(k-3)}{2}$ for all $0 \leq k \leq n-1$. Then by the recurrence relation (3.2) we have

$$\text{Tdeg}(\tilde{\beta}_n) = \text{Tdeg}([n+1]_q \tilde{\beta}_n) - n = \text{Tdeg} \left(\sum_{k=0}^{n-1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \tilde{\beta}_k \right) - n.$$

By the inductive hypothesis,

$$\text{Tdeg} \left(\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \tilde{\beta}_k \right) = k(n+1-k) + \frac{k(k-3)}{2} = -\frac{1}{2}k^2 + \left(n - \frac{1}{2}\right)k$$

which is an increasing function of k for $0 \leq k \leq n-1$. Thus Proposition 4.4(b) applies and we find that

$$\begin{aligned} \text{Tdeg}(\tilde{\beta}_n) &= \max_{0 \leq k \leq n-1} \left(-\frac{1}{2}k^2 + \left(n - \frac{1}{2}\right)k \right) - n \\ &= -\frac{1}{2}(n-1)^2 + \left(n - \frac{1}{2}\right)(n-1) - n = \frac{n(n-3)}{2}. \end{aligned}$$

□

Since it is well-known that the odd indexed Bernoulli numbers $B_{2n+1} = 0$ for $n \geq 1$, we expect that $\tilde{\beta}_{2n+1}$ has a zero at $q = 1$ for $n \geq 1$. Looking at the list of the $\tilde{\beta}_n$ for $n > 1$, we observe that the zero at $q = 1$ is simple, and that there is also a zero of order n at $q = 0$ for all $n > 1$. We now prove these facts by taking limits with the generating function.

Theorem 4.4. *The $\tilde{\beta}_n$ have a zero of order exactly n at $q = 0$ except at $n = 1$.*

Proof. Recall the generating function

$$F(t, q) = \sum_{n \geq 0} \tilde{\beta}_n \frac{t^n}{(q; q)_n} = \frac{t}{1-q} \frac{(t; q)_\infty}{1 - (t; q)_\infty}.$$

Replacing t by t/q we find that

$$\begin{aligned} \tilde{\beta}_0 + \tilde{\beta}_1 \frac{t}{q(1-q)} + \sum_{n \geq 2} \frac{\tilde{\beta}_n}{q^n} \frac{t^n}{(q; q)_n} &= 1 - \frac{t}{q(1-q^2)} + \sum_{n \geq 2} \frac{\tilde{\beta}_n}{q^n} \frac{t^n}{(q; q)_n} \\ &= \frac{t}{q(1-q)} \frac{(1-tq^{-1})(t; q)_\infty}{1 - (1-tq^{-1})(t; q)_\infty}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{n \geq 2} \frac{\tilde{\beta}_n}{q^n} \frac{t^n}{(q; q)_n} \\ &= \frac{t}{q(1-q)} \frac{(1-tq^{-1})(t; q)_\infty}{1 - (1-tq^{-1})(t; q)_\infty} - 1 + \frac{t}{q(1-q^2)} \\ &= \frac{t}{1-q^2} \left(\frac{(tq^{-1}; q)_\infty(1+q) - t^{-1}q(1-q^2)(1 - (tq^{-1}; q)_\infty) + (1 - (tq^{-1}; q)_\infty)}{q - (q-t)(tq; q)_\infty} \right) \\ (4.8) \quad &= \frac{t}{1-q^2} \left(\frac{(tq^{-1}; q)_\infty (q + t^{-1}q(1-q^2)) + 1 - t^{-1}q(1-q^2)}{q - (q-t)(t; q)_\infty} \right). \end{aligned}$$

To prove that the order is exactly n , use the fact that

$$\lim_{q \rightarrow 0^+} q(tq^{-1}; q)_\infty = -t(1-t)$$

to obtain

$$\lim_{q \rightarrow 0^-} \sum_{n \geq 2} \frac{\tilde{\beta}_n}{q^n} \frac{t^n}{(q; q)_n} = t \left(\frac{-t(1-t) + t^{-1}(-t)(1-t) + 1}{t(1-t)} \right) = -1 - t + \frac{1}{1-t}.$$

We have thus proven a slightly stronger statement than required, that is, $\lim_{q \rightarrow 0} \tilde{\beta}_n/q^n = 1$, for all $n \geq 2$. \square

To prove that the $\tilde{\beta}_{2n+1}$ have a simple zero at $q = 1$, we will also prove a stronger statement than required. In fact, the statement we will prove will provide a connection to *even* Bernoulli numbers. To isolate the $\tilde{\beta}_{2n+1}$ in the generating function, we write

$$\begin{aligned} \sum_{n=1}^{\infty} \tilde{\beta}_{2n+1} \frac{t^n}{(q; q)_n} &= \frac{1}{2} [F(t, q) - F(-t, q)] - \tilde{\beta}_1 \frac{t}{1-q} \\ &= \frac{1}{2} \cdot \frac{t}{1-q} \left(\left(\frac{1}{(t; q)_\infty} - 1 \right)^{-1} + \left(\frac{1}{(-t; q)_\infty} - 1 \right)^{-1} + \frac{2}{1+q} \right). \end{aligned}$$

Therefore,

$$(4.9) \quad \sum_{n=1}^{\infty} \lim_{q \rightarrow 1^-} \left(\frac{\tilde{\beta}_{2n+1}}{1-q} \right) \frac{t^{2n+1}}{(2n+1)!} = \lim_{q \rightarrow 1^-} \left(\frac{t}{2(1-q)} \cdot N(q) \right),$$

with

$$(4.10) \quad N(q) := \left(\frac{1}{(t(1-q); q)_\infty} - 1 \right)^{-1} + \left(\frac{1}{(-t(1-q); q)_\infty} - 1 \right)^{-1} + \frac{2}{1+q}.$$

Note that

$$\lim_{q \rightarrow 1^-} N(q) = \frac{1}{e^t - 1} + \frac{1}{e^{-t} - 1} + 1 = 0,$$

and so by l'Hôpital's Rule

$$(4.11) \quad \lim_{q \rightarrow 1^-} \left(\frac{t}{2(1-q)} \cdot N(q) \right) = -\frac{t}{2} N'(1),$$

provided $N'(1)$ exists and is finite (as a left-sided derivative).

In evaluating $N'(1)$, it is convenient to separately evaluate the derivative of a related object as a lemma.

Lemma 4.1.

$$(4.12) \quad \frac{1}{(t(1-q); q)_\infty} \cdot \frac{d}{dq} ((t(1-q); q)_\infty) = \sum_{n=1}^{\infty} \frac{(1-q)^n (1-q^n)}{(1-q^{n+1})^2} t^{n+1}$$

Proof. The left-hand side is the logarithmic derivative

$$\begin{aligned} \frac{d}{dq} (\log(t(1-q); q)_\infty) &= \sum_{k=0}^{\infty} \frac{\frac{d}{dq} (1 - q^k(1-q)t)}{1 - q^k(1-q)t} \\ &= \sum_{k=1}^{\infty} \frac{-kq^{k-1}t}{1 - q^k(1-q)t} + \sum_{k=0}^{\infty} \frac{(k+1)q^k t}{1 - q^k(1-q)t} \\ &= \sum_{k=1}^{\infty} kq^{k-1}t \left(\frac{-1}{1 - q^k(1-q)t} + \frac{1}{1 - q^{k-1}(1-q)t} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} k q^{k-1} t \sum_{n=1}^{\infty} (q^{nk-n} - q^{(nk-n)+n})(1-q)^n t^n \\
&= \sum_{n=1}^{\infty} (1-q)^n (1-q^n) t^{n+1} \sum_{k=1}^{\infty} k (q^{n+1})^{k-1} \\
&= \sum_{n=1}^{\infty} \frac{(1-q)^n (1-q^n)}{(1-q^{n+1})^2} t^{n+1}.
\end{aligned}$$

□

Now we state and prove the theorem on zeroes at $q = 1$.

Theorem 4.5. *For all $n \geq 1$,*

$$\lim_{q \rightarrow 1^-} \left(\frac{\tilde{\beta}_{2n+1}}{1-q} \right) = \left(n^2 - \frac{1}{4} \right) B_{2n}.$$

Proof. Using (4.10) and (4.12),

$$\begin{aligned}
\lim_{q \rightarrow 1^-} N'(q) &= \lim_{q \rightarrow 1^-} \left[\frac{((t(1-q); q)_{\infty})^{-1}}{(((t(1-q); q)_{\infty})^{-1} - 1)^2} \cdot \sum_{n=1}^{\infty} \frac{(1-q)^n (1-q^n)}{(1-q^{n+1})^2} t^{n+1} \right. \\
&\quad \left. + \frac{((-t(1-q); q)_{\infty})^{-1}}{(((t(1-q); q)_{\infty})^{-1} - 1)^2} \cdot \sum_{n=1}^{\infty} \frac{(1-q)^n (1-q^n)}{(1-q^{n+1})^2} (-t)^{n+1} - \frac{2}{(1+q)^2} \right].
\end{aligned}$$

In each of the infinite sums above, because of the factor of $(1-q)^n$ in the numerator, all but the $n = 1$ terms will vanish in the limit. Therefore, our one-sided derivative exists:

$$(4.13) \quad N'(1) = \frac{e^t}{(e^t - 1)^2} \cdot \frac{t^2}{4} + \frac{e^{-t}}{(e^{-t} - 1)^2} \cdot \frac{t^2}{4} - \frac{1}{2} = \frac{t^2}{4} \left(\frac{2e^t}{(e^t - 1)^2} \right) - \frac{1}{2}.$$

Using (4.13), along with (4.9) and (4.11), we get:

$$\begin{aligned}
\sum_{n=1}^{\infty} \lim_{q \rightarrow 1^-} \left(\frac{\tilde{\beta}_{2n+1}}{1-q} \right) \frac{t^{2n+1}}{(2n+1)!} &= -\frac{t}{4} \left(\frac{t^2 e^t}{(e^t - 1)^2} - 1 \right) \\
&= \frac{t^3}{4} \left(\frac{1}{t^2} - \frac{e^t(e^t - 1) - e^t(e^t + 1)}{2(e^t - 1)^2} \right) \\
&= \frac{t^3}{4} \cdot \frac{d}{dt} \left(-\frac{1}{t} + \frac{1}{e^t - 1} + \frac{1}{2} \right) \\
&= \frac{t^3}{4} \cdot \frac{d}{dt} \left(\frac{1}{t} \sum_{n=1}^{\infty} B_{2n} \frac{t^{2n}}{(2n)!} \right) \\
&= \sum_{n=1}^{\infty} \frac{B_{2n}(2n-1)(2n+1)}{4} \frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}$$

In the step where we expand in a series, we appeal to fact that the expanded function is even. □

Aside from being interesting in its own right, because $B_{2n} \neq 0$ for all $n \geq 1$, this theorem has our desired result as an immediate corollary.

Corollary 4.2. *For all $n \geq 1$, $\tilde{\beta}_{2n+1}$ has a zero of order 1 at $q = 1$.*

This also constitutes a point of similarity with Carlitz's q -Bernoulli numbers. Both the previous theorem and its corollary are similar to properties of the β_n [3, Sec. 7, Thm. (iii)].

We close this section with a theorem on the poles of $\tilde{\beta}_n$.

Theorem 4.6. *For all $n \geq 1$, $\tilde{\beta}_n$ has simple poles at the primitive $n + 1$ st roots of unity.*

Proof. Fix $n \geq 1$. We apply the explicit formula (4.2) to find that the term corresponding to the singleton partition (n) gives rise to a term $-1/[n + 1]_q$ in the sum. In the remaining partitions all the parts are less than n and so $k_i + 1 \leq n$ in the rest of the summands. Thus, cancelling the $[2n]_q!$ we see that the summands are all of the form

$$(-1)^{\ell(p)} NC(p) \frac{[n]_q!}{[k_1 + 1]_q! \cdots [k_{\ell(p)} + 1]_q!}$$

which are analytic at primitive $n + 1$ st roots of unity. Thus we have simple poles at ζ_{n+1} arising from the $-1/[n + 1]_q$ term. \square

5. q -POWERS AND q -BERNOULLI POLYNOMIALS

One of the most important classical applications of the Bernoulli numbers is the power sum formula, which involves the Bernoulli polynomials $B_n(x)$ given by

$$(5.1) \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k},$$

and satisfying the generating function

$$(5.2) \quad \sum_{n \geq 0} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}.$$

The power sum formula can then be expressed as [5, Eq. 24.4.7]

$$(5.3) \quad S_n(m) = \frac{1}{n + 1} (B_{n+1}(m) - B_{n+1}), \text{ where } S_n(m) := \sum_{k=1}^{m-1} k^n,$$

valid for all positive integers m and n . In this section we show that our sequence $\tilde{\beta}_n$ satisfy an analogue of (5.3), with an appropriate q -analogue of the Bernoulli polynomials as well as that of an n th power. Let us first state the result.

Theorem 5.1. *For all integers $n, x \geq 1$, we have*

$$\tilde{\sigma}_n(x) = \frac{1}{[n + 1]_q} (\tilde{\beta}_{n+1}(x) - \tilde{\beta}_{n+1}),$$

where

$$\begin{aligned} \tilde{\sigma}_n(x) &:= \sum_{k=1}^{x-1} k_q^{(n)}, \\ \tilde{\beta}_n(x) &:= \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q \tilde{\beta}_k x_q^{(n-k)}, \end{aligned}$$

and

$$k_q^{(n)} := \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + i_2 + \dots + i_k = n}} \left[\begin{matrix} n \\ i_1, i_2, \dots, i_k \end{matrix} \right]_q.$$

Remark 5.2. It is clear from the multinomial theorem that $\lim_{q \rightarrow 1} k_q^{(n)} = k^n$ for all positive integers k , and therefore Theorem 5.1 indeed specializes to (5.3) in the limit.

Before we prove Theorem 5.1, we need to first derive the generating functions for $k_q^{(n)}$ and $\tilde{\beta}_n(x)$.

Proposition 5.1. *For all integers $k \geq 1$,*

$$(5.4) \quad \sum_{n \geq 0} k_q^{(n)} \frac{t^n}{(q; q)_n} = \left(\frac{1}{(t; q)_\infty} \right)^k.$$

Proof. Expanding the right-hand side by the q -binomial theorem (1.4) we find that

$$\begin{aligned} \left(\frac{1}{(t; q)_\infty} \right)^k &= \left(\sum_{i_1 \geq 0} \frac{t^{i_1}}{(q; q)_{i_1}} \right) \cdots \left(\sum_{i_k \geq 0} \frac{t^{i_k}}{(q; q)_{i_k}} \right) \\ &= \sum_{n \geq 0} \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \dots + i_k = n}} \frac{(q; q)_n}{(q; q)_{i_1} \cdots (q; q)_{i_k}} \cdot \frac{t^n}{(q; q)_n} \\ &= \sum_{n \geq 0} k_q^{(n)} \frac{t^n}{(q; q)_n}. \end{aligned}$$

□

Using (5.4) we may in fact extend the definition of $k_q^{(n)}$ to all real numbers k . This observation justifies the name *q -Bernoulli polynomial* for $\tilde{\beta}_n(x)$. The next proposition gives us the generating function for $\tilde{\beta}_n(x)$.

Proposition 5.2. *For $x \in \mathbb{R}$ we have*

$$(5.5) \quad \sum_{n \geq 0} \tilde{\beta}_n(x) \frac{t^n}{(q; q)_n} = \frac{t}{1-q} \cdot \frac{\left(\frac{1}{(t; q)_\infty} \right)^x}{\left(\frac{1}{(t; q)_\infty} - 1 \right)}.$$

Proof. We apply the definition of $\tilde{\beta}_n(x)$ to the generating function and reindex (replace n by $n+k$) to find

$$\begin{aligned} \sum_{n \geq 0} \tilde{\beta}_n(x) \frac{t^n}{(q; q)_n} &= \sum_{n \geq 0} \frac{t^n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\beta}_k x_q^{(n-k)}, \\ &= \sum_{n \geq 0} x_q^{(n)} \frac{t^n}{(q; q)_n} \sum_{k \geq 0} \tilde{\beta}_k \frac{t^k}{(q; q)_k} \\ &= \left(\frac{1}{(t; q)_\infty} \right)^x \cdot \frac{t}{1-q} \left(\frac{1}{\left(\frac{1}{(t; q)_\infty} - 1 \right)} \right), \end{aligned}$$

as desired. □

We note that, as usual, upon replacing t by $t(1-q)$ and taking the limit as q tends to 1, we recover the classical generating function (5.2). We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $x \in \mathbb{N}$. We construct the generating function for the right-hand side (recall $n \geq 1$) and simplify

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{[n+1]_q} \left(\tilde{\beta}_{n+1}(x) - \tilde{\beta}_{n+1} \right) \frac{t^n}{(q; q)_n} &= \frac{1-q}{t} \sum_{n \geq 1} \left(\tilde{\beta}_{n+1}(x) - \tilde{\beta}_{n+1} \right) \frac{t^{n+1}}{(q; q)_{n+1}} \\ &= \frac{1-q}{t} \sum_{n \geq 0} \left(\tilde{\beta}_n(x) - \tilde{\beta}_n \right) \frac{t^n}{(q; q)_n} \end{aligned}$$

where we used the fact that $\tilde{\beta}_0(x) = \tilde{\beta}_0 = 1$ in the last equality. Applying (5.5), (3.1), and (5.4) we obtain

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{[n+1]_q} \left(\tilde{\beta}_{n+1}(x) - \tilde{\beta}_{n+1} \right) \frac{t^n}{(q; q)_n} &= \frac{\left(\frac{1}{(t; q)_\infty} \right)^x - 1}{\frac{1}{(t; q)_\infty} - 1} = \sum_{k=0}^{x-1} \left(\frac{1}{(t; q)_\infty} \right)^k \\ &= \sum_{k=0}^{x-1} \sum_{n \geq 0} k_q^{(n)} \frac{t^n}{(q; q)_n} = \sum_{n \geq 0} \tilde{\sigma}_n(x) \frac{t^n}{(q; q)_n}, \end{aligned}$$

as required. \square

6. q -STIRLING NUMBERS

The q -analogue of n th powers $k_q^{(n)}$ defined in the previous section opens the door to many extensions of classical identities to our world of $\tilde{\beta}_n$. In this section, we prove a generalization of [6, Eq. 24.6.9]:

$$(6.1) \quad B_n = \sum_{j=0}^n \frac{(-1)^j j!}{j+1} S(n, j) = \sum_{j=0}^n \frac{1}{j+1} \sum_{k=0}^j (-1)^k \binom{j}{k} k^n,$$

where $S(n, j)$ are the *Stirling numbers of the second kind*. They satisfy the generating function

$$(6.2) \quad x^n = \sum_{k=0}^n x(x-1) \cdots (x-k+1) S(n, k),$$

and closed form

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

To prove our generalization of (6.1), we define $\tilde{S}(n, k)$, a q -analogue of $S(n, k)$, based on $x_q^{(n)}$ via the explicit formula above. Let

$$(6.3) \quad \tilde{S}(n, k) := \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j_q^{(n)},$$

and $\tilde{S}(0, 0) := 1$.

(6.3) may be compared with the natural analogue within the Carlitz framework in [3], given by

$$a_{n,k} := \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \left[\begin{matrix} k \\ j \end{matrix} \right]_q ([k-j]_q)^n.$$

The $a_{n,j}$ appear in a closed form for the β_n , replacing $S(n, j)$ in Carlitz's extension of (6.1). They were further generalized by Gould [10]. An overview is presented in [9], where identities concerning q -extensions of both Bernoulli and Stirling numbers are generated via q -difference equations.

Our q -Stirling numbers $\tilde{S}(n, k)$ satisfy an analogue of (6.2):

Proposition 6.1. *For all nonnegative integers n and fixed real x ,*

$$(6.4) \quad x_q^{(n)} = \sum_{k=0}^n (x)(x-1) \cdots (x-k+1) \tilde{S}(n, k).$$

Proof. By (5.4) and the binomial expansion

$$(6.5) \quad y^x = ((y-1) + 1)^x = \sum_{k \geq 0} \binom{x}{k} (y-1)^k,$$

with $y = \frac{1}{(t; q)_\infty}$, we have

$$\sum_{n \geq 0} x_q^{(n)} \frac{t^n}{(q; q)_n} = \left(\frac{1}{(t; q)_\infty} \right)^x = \sum_{k \geq 0} \binom{x}{k} \left(\frac{1}{(t; q)_\infty} - 1 \right)^k,$$

valid on sufficiently small discs $0 < |q|, |t| < \varepsilon$ for any real x . Now we expand the k th power and applying (5.4) again, we obtain

$$\begin{aligned} \sum_{n \geq 0} x_q^{(n)} \frac{t^n}{(q; q)_n} &= \sum_{k \geq 0} \binom{x}{k} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \left(\frac{1}{(t; q)_\infty} \right)^j \\ &= \sum_{k \geq 0} \binom{x}{k} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{n \geq 0} j_q^{(n)} \frac{t^n}{(q; q)_n}. \end{aligned}$$

However, the innermost sum actually begins at $n = k$ since the lowest power of t in the expansion of $\left(\frac{1}{(t; q)_\infty} - 1 \right)^k$ is t^k . Therefore,

$$\begin{aligned} \sum_{n \geq 0} x_q^{(n)} \frac{t^n}{(q; q)_n} &= \sum_{k \geq 0} \binom{x}{k} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{n \geq k} j_q^{(n)} \frac{t^n}{(q; q)_n} \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j_q^{(n)} \right) \frac{t^n}{(q; q)_n} \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^n (x)(x-1) \cdots (x-k+1) \tilde{S}(n, k) \right) \frac{t^n}{(q; q)_n}. \end{aligned}$$

The result follows by equating coefficients of $t^n/(q; q)_n$ and analytically continuing the identity to the entire unit circle $|q| < 1$. \square

Hidden in the above proof is the fact that for fixed k the generating function for $\tilde{S}(n, k)$ may be expressed as

$$\sum_{n \geq 0} \tilde{S}(n, k) \frac{t^n}{(q; q)_n} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \left(\frac{1}{(t; q)_\infty} \right)^k = \frac{1}{k!} \left(\frac{1}{(t; q)_\infty} - 1 \right)^k.$$

Since the power series expansion of this function begins at t^k , we have the following result.

Proposition 6.2. *For all integers $n \geq 0$, if $k > n$ then $\tilde{S}(n, k) = 0$.*

Since $S(n, k) = 0$ for $k > n$, we would at least expect our q -analogue to vanish under the limit $q \rightarrow 1$. It is somewhat surprising that as functions of q they are identically zero. However, Carlitz's q -Stirling numbers $a_{n,k}$ also vanish for $k > n$ [3].

To obtain the desired generalization of (6.1), we first prove an evaluation of the q -analogue of the middle expression in (6.1) in terms of a sum involving $\tilde{\beta}_n$. This will eventually lead to the closed form for $\tilde{\beta}_n$ we are after.

Lemma 6.1. *For all $n \in \mathbb{N}$, we have*

$$(6.6) \quad \sum_{j=0}^n \frac{(-1)^j j!}{j+1} \tilde{S}(n, j) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\beta}_{n-k} \frac{(q; q)_k}{[k+1]_q (k+1)}.$$

Proof. We begin with the generating function

$$(6.7) \quad T(x, t, q) := \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \frac{\tilde{S}(n, j)}{j+1} x(x-1) \cdots (x-j) \right) \frac{t^n}{(q; q)_n}.$$

By Proposition 6.2, the inner sum can be extended to infinity. Applying (6.3) and exchanging the order of summation, we have

$$\begin{aligned} T(x, t, q) &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(j+1)j!} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} m_q^{(n)} x(x-1) \cdots (x-j) \frac{t^n}{(q; q)_n} \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \binom{x}{j+1} \sum_{n=0}^{\infty} \frac{m_q^{(n)} t^n}{(q; q)_n} \\ &= \sum_{j=0}^{\infty} \binom{x}{j+1} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \left(\frac{1}{(t; q)_{\infty}} \right)^m \\ &= \sum_{j=0}^{\infty} \binom{x}{j+1} \left(\frac{1}{(t; q)_{\infty}} - 1 \right)^j = \frac{1}{\frac{1}{(t; q)_{\infty}} - 1} \sum_{j=1}^{\infty} \binom{x}{j} \left(\frac{1}{(t; q)_{\infty}} - 1 \right)^j. \end{aligned}$$

Add and subtract the $j = 0$ term for this sum, and then use the binomial expansion (6.5) again to obtain

$$T(x, t, q) = \frac{\left(\frac{1}{(t; q)_{\infty}} \right)^x - 1}{\frac{1}{(t; q)_{\infty}} - 1}.$$

Next we divide by x and take the limit $x \rightarrow 0$. By l'Hôpital's Rule,

$$(6.8) \quad \lim_{x \rightarrow 0} \frac{T(x, t, q)}{x} = \frac{\ln\left(\frac{1}{(t; q)_{\infty}}\right)}{\frac{1}{(t; q)_{\infty}} - 1}.$$

Expanding the numerator in a series, we get

$$\begin{aligned} \ln\left(\frac{1}{(t; q)_{\infty}}\right) &= -\sum_{k=0}^{\infty} \ln(1 - q^k t) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{(q^k t)^j}{j} = \sum_{j=1}^{\infty} \frac{t^j}{j} \sum_{k=0}^{\infty} (q^j)^k \\ &= \sum_{j=1}^{\infty} \frac{t^j}{j(1 - q^j)} = \frac{t}{(1 - q)} \cdot \sum_{j=0}^{\infty} \frac{t^j}{(j+1)[j+1]_q}. \end{aligned}$$

Combine this with (6.8) and (3.1) to get

$$\lim_{x \rightarrow 0} \frac{T(x, t, q)}{x} = \sum_{j=0}^{\infty} \frac{t^j}{(j+1)[j+1]_q} \cdot F(t, q).$$

Then, we multiply power series on the right, and combine the result with (6.7) and (3.1), to get the desired formula. \square

We now systematically solve (6.6) for $\tilde{\beta}_n$. The first step is to get rid of $\tilde{\beta}_{n-1}$. Isolating $\tilde{\beta}_n$ and separating the $\tilde{\beta}_{n-1}$ term from the sum gives

$$(6.9) \quad \tilde{\beta}_n = \sum_{j=0}^n \frac{(-1)^j j!}{j+1} \tilde{S}(n, j) - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \tilde{\beta}_{n-1} \frac{(q; q)_1}{[2]_q (2)} - \sum_{k=2}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\beta}_{n-k} \frac{(q; q)_k}{[k+1]_q (k+1)}.$$

Substitute $\tilde{\beta}_{n-1}$ by using (6.9), with n replaced by $n-1$, to get

$$\begin{aligned}\tilde{\beta}_n &= \sum_{j=0}^n \frac{(-1)^j j!}{j+1} \tilde{S}(n, j) - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \sum_{j=0}^{n-1} \frac{(-1)^j j!}{j+1} \tilde{S}(n-1, j) - \sum_{k=2}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\beta}_{n-k} \frac{(q; q)_k}{[k+1]_q(k+1)} \\ &+ \begin{bmatrix} n \\ 1 \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tilde{\beta}_{n-1-k} \frac{(q; q)_k}{[k+1]_q(k+1)}.\end{aligned}$$

Reindex the sum on the second line and combine it with the other sum over k , completing the first step of the manipulation:

$$\begin{aligned}\tilde{\beta}_n &= \sum_{j=0}^n \frac{(-1)^j j!}{j+1} \tilde{S}(n, j) + \left(- \begin{bmatrix} n \\ 1 \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \right) \sum_{j=0}^{n-1} \frac{(-1)^j j!}{j+1} \tilde{S}(n-1, j) \\ &+ \sum_{k=2}^n \tilde{\beta}_{n-k} \left(- \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q; q)_k}{[k+1]_q(k+1)} + \begin{bmatrix} n \\ 1, k-1, n-k \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \frac{(q; q)_{k-1}}{[k]_q(k)} \right).\end{aligned}$$

Next, we remove β_{n-2} from the formula, by using (6.9) again, this time with n replaced by $n-2$. The result of this step is

$$\begin{aligned}\tilde{\beta}_n &= \sum_{j=0}^n \frac{(-1)^j j!}{j+1} \tilde{S}(n, j) + \left(- \begin{bmatrix} n \\ 1 \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \right) \sum_{j=0}^{n-1} \frac{(-1)^j j!}{j+1} \tilde{S}(n-1, j) \\ &+ \left(- \begin{bmatrix} n \\ 2 \end{bmatrix}_q \frac{(q; q)_2}{[3]_q(3)} + \begin{bmatrix} n \\ 1, 1, n-2 \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \frac{(q; q)_1}{[2]_q(2)} \right) \sum_{j=0}^{n-2} \frac{(-1)^j j!}{j+1} \tilde{S}(n-2, j) \\ &+ \sum_{k=3}^n \tilde{\beta}_{n-k} \left(- \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q; q)_k}{[k+1]_q(k+1)} + \begin{bmatrix} n \\ 1, k-1, n-k \end{bmatrix}_q \frac{(q; q)_1}{[2]_q(2)} \frac{(q; q)_{k-1}}{[k]_q(k)} \right. \\ &\left. + \begin{bmatrix} n \\ 2, k-2, n-k \end{bmatrix}_q \frac{(q; q)_2}{[3]_q(3)} \frac{(q; q)_{k-2}}{[k-1]_q(k-1)} - \begin{bmatrix} n \\ 1, 1, k-1, n-k \end{bmatrix}_q \left(\frac{(q; q)_1}{[2]_q(2)} \right)^2 \frac{(q; q)_{k-1}}{[k]_q(k)} \right).\end{aligned}$$

Continuing in this way, we eliminate the sum indexed by k , which leaves

$$\tilde{\beta}_n = \sum_{m=0}^n C_{m,n} \sum_{j=0}^{n-m} \frac{(-1)^j j!}{j+1} \tilde{S}(n-m, j)$$

where

$$C_{m,n} := \sum_{p=(i_1, i_2, \dots, i_s) \in \mathcal{C}(m)} (-1)^s \begin{bmatrix} n \\ i_1, i_2, \dots, i_s, n-m \end{bmatrix}_q \cdot \prod_{\ell=1}^s \frac{(q; q)_{i_\ell}}{[i_\ell+1]_q(i_\ell+1)}.$$

The q -multinomial in the coefficients $C_{m,n}$ can be cancelled with part of the product, giving us

$$C_{m,n} = \sum_{p=(i_1, i_2, \dots, i_s) \in \mathcal{C}(m)} (-1)^s \frac{(q; q)_n}{(q; q)_{n-m}} \cdot \prod_{\ell=1}^s \frac{1}{[i_\ell+1]_q(i_\ell+1)}.$$

Notice that one factor inside the sum does not depend on p , and so can be factored outside the sum. The remaining part is independent of n . Thus we obtain the following theorem.

Theorem 6.1. (Closed form for $\tilde{\beta}_n$) For all $n \in \mathbb{N}$,

$$\tilde{\beta}_n = \sum_{m=0}^n (q^{n-m+1}; q)_m c_m \sum_{j=0}^{n-m} \frac{(-1)^j j!}{j+1} \tilde{S}(n-m, j)$$

$$= \sum_{m=0}^n (q^{n-m+1}; q)_m c_m \sum_{j=0}^{n-m} \frac{1}{j+1} \sum_{k=0}^j (-1)^k \binom{j}{k} k_q^{(n-m)},$$

where

$$c_m := \sum_{p=(i_1, i_2, \dots, i_s) \in \mathcal{C}(m)} \prod_{\ell=1}^s \frac{-1}{[i_\ell + 1]_q (i_\ell + 1)} \text{ for } m \geq 0.$$

Remark 6.2. We remark that the factor $(q^{n-m+1}; q)_m \rightarrow 0$ as $q \rightarrow 1$, except when $m = 0$. Hence, this closed form is indeed a q -analogue (6.1). This is another point of similarity to Carlitz's q -analogue; the β_n also admit a formula relating them to the $a_{n,k}$, leading to closed form which is a double sum [3, (6.3)]. Even though our closed form is a triple sum, the ‘‘additional’’ sum (c_m) reveals yet another combinatorial connection to integer compositions distinct from that in (4.1), our analogue of Woon's Theorem.

7. CONCLUDING REMARKS

Our motivation here was to apply the unified framework of Strodt operators from [2] to develop a natural set of q -analogues to the Bernoulli numbers. More generally, we may define the q -moment generating function of a distribution function $g(x)$ will be given as

$$Q(t) := \sum_{n=0}^{\infty} \mu_n \frac{t^n}{(q; q)_n} = \int_{-\infty}^{\infty} \frac{g(x)}{(xt; q)_{\infty}} d_q x$$

The integral is a definite integral over the entire real line, where the bounds on the interval of integration may be effectively limited by the support of the distribution function.

Under this framework, the generating function for the closely related q -Euler numbers would thus be

$$E(t) := \sum_{n=0}^{\infty} E_n \frac{t^n}{(q; q)_n} = \frac{1}{Q(t)},$$

where $Q(t)$ is the q -moment generating function for the distribution corresponding to the Euler numbers: the average of a pair of Dirac delta functions at -1 and 1 (n. b.: In [2], the authors used 0 and 1 . However, this generated *scaled* Euler numbers, whereas using the values -1 and 1 produces the numbers themselves.).

$$Q(t) = \int_{-\infty}^{\infty} \frac{\delta_{-1}(x) + \delta_1(x)}{2(xt, q)_{\infty}} d_q x = \frac{1}{2(-t; q)_{\infty}} + \frac{1}{2(t; q)_{\infty}}$$

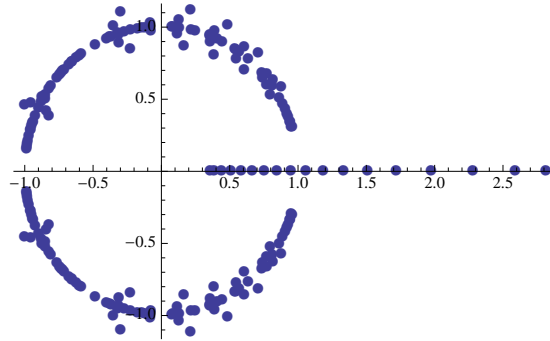
so that

$$E(t) = \frac{1}{\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} \frac{t^n}{(q; q)_n}} = \frac{2(t^2; q^2)_{\infty}}{(-t; q)_{\infty} + (t; q)_{\infty}}.$$

This definition for q -Euler numbers is similar to that used in [11], where the authors provide a combinatorial interpretation for their Taylor coefficients as power series in q , in terms of alternating permutations. It would be interesting to provide a similar interpretation for the coefficients of $\tilde{\beta}_n$ as Taylor series in the variable q .

We have also shown that the $\tilde{\beta}_n$ admit similar properties and closed forms to the major properties and closed forms of the Carlitz sequence β_n . The only major property of the β_n that appears in [3] which we have yet to prove is a (weaker) q -version of the von Staudt-Clausen Theorem. It is possible that, using one of our closed forms, a similar theorem can be proven for the $\tilde{\beta}_n$.

We have provided only a very brief investigation into the analytic properties of $\tilde{\beta}_n$. Below is a plot of the zeros of $\tilde{\beta}_{20}/q^{20}$ on the complex plane, whose shape is typical of general plots for the zeroes of $\tilde{\beta}_n$.



From this plot and others, as well as explicit expressions for $\tilde{\beta}_n$, we see there is a lot of structure still to be proven. For example, the complex zeroes seem to cluster around the unit circle, with gaps (repulsion) near the $n + 1$ st roots of unity. We close with a collection of our observations in the following conjecture.

Conjecture 7.1. *For all $n \geq 1$ we have*

- (a) $\tilde{\beta}_n$ has no zeroes on the negative real line.
- (b) $\tilde{\beta}_n/q^n$ is self-reciprocal for $n \geq 2$. That is, $q^{n(n-1)/2}\tilde{\beta}_n(1/q) = \tilde{\beta}_n(q)$. This means that for every zero inside the unit circle there is a corresponding zero with the same multiplicity outside.
- (c) The zeroes of $\tilde{\beta}_n/q^n$ of largest and smallest modulus are real.
- (d) There is some small, positive function $f(n)$ such that the complex zeroes $|z|$ of $\tilde{\beta}_n$ satisfy $1 - f(n) < |z|$.

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E-mail address: `math@oyeat.com`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, VIRGINIA WESLEYAN COLLEGE, NORFOLK, VA 23502
USA

E-mail address: `dantemanna@gmail.com`