Higher-dimensional box integrals

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Abstract. Herein, with the aid of substantial symbolic computation, we solve previously open problems in the theory of n-dimensional box integrals $B_n(s) := \langle |\vec{r}|^s \rangle$; $\vec{r} \in [0,1]^n$. In particular we resolve an elusive integral called $\mathcal{K}_5$ that previously acted as a "blockade" against closed-form evaluation in $n = 5$ dimensions. In consequence we now know that $B_n(\text{integer})$ can be given a closed form for $n = 1, 2, 3, 4, 5$. We also find the general residue at the pole at $s = -n$, this leading to new relations and definite integrals—for example, we are able to give the first nontrivial closed forms for 6-dimensional box integrals and to show hyperclosure of $B_6(\text{even})$. The Clausen function and its generalizations play a central role in these higher-dimensional evaluations. Our results provide stringent test scenarios for symbolic-algebra simplification methods.

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1 Background and Nomenclature

Recent papers by Bailey, Borwein, and Crandall [5, 6]—stemming from the historical work of Anderssen et al. [1]—have exhibited new results on the computation and analysis of *box integrals*, these being expectations of radius and separation, respectively, within an n-cube. We write formally $B_n(s) := \langle |\vec{r}|^s \rangle$, $\Delta_n(s) := \langle |\vec{r}-\vec{q}|^s \rangle$, with specific definitions

$$\begin{aligned} B_n(s) &:= \int_{\vec{r}\in[0,1]^n} |\vec{r}|^s \; \mathcal{D}\vec{r} \qquad (1.1)\\ &= \int_0^1 \cdots \int_0^1 \left(r_1^2+\cdots+r_n^2\right)^{s/2} \; dr_1\cdots dr_n, \\ \Delta_n(s) &:= \int_{\vec{r},\vec{q}\in[0,1]^n} |\vec{r}-\vec{q}|^s \; \mathcal{D}\vec{r}\;\mathcal{D}\vec{q} \qquad (1.2)\\ &= \int_0^1 \cdots \int_0^1 \left((r_1-q_1)^2+\cdots+(r_n-q_n)^2\right)^{s/2} \; dr_1\cdots dr_n \; dq_1\cdots dq_n. \end{aligned}$$

Introduced in the work [6] is a useful function, a kind of generalized box integral:

$$C_{m,0}(s,a) \;:=\; \int_{\vec{r}\in[0,1]^m} (a+r^2)^{s/2} \; \mathcal{D}\vec{r}. \qquad (1.3)$$

Let us list some relevant known facts and interrelations amongst these functions. It is important to observe that $B_n(s), \Delta_n(s)$ have well defined analytic continuations over the entire complex s-plane, with $B_n(s)$ having a solitary pole at $s=-n$ and $\Delta_n(s)$ having exactly $n+1$ poles, at $s = -2n, -2n+1, \ldots, -n$. In particular, though something like $B_4(-5)$ does not converge as a literal box integral, its value exists unambiguously, and is known in closed form, namely as a *negative* value $-\sqrt{8}\arctan(1/\sqrt{8})$. References [5, 6] establish the analytic properties together with the following results:

1. **B, C relations:**

$$\begin{aligned} B_n(s) &= \frac{n}{n+s} C_{n-1,0}(s,1), \qquad (1.4)\\ B_n(-n-1) &= -nC_{n-2,0}(1-n,2), \\ \mathrm{Res}_n &= nC_{n-1}(-n,1), \qquad (1.5) \end{aligned}$$

where $\mathrm{Res}_n := \lim_{\epsilon\to 0} \epsilon B_n(-n+\epsilon)$ is the residue of B_n at the solitary pole ($s=-n$).

2. **Absolutely convergent analytic series for B, with pole at $s=-n$:**

$$B_n(s) \;=\; \frac{n^{1+s/2}}{s+n} \sum_{k\geq 0} \gamma_{n-1,k} \left(\frac{2}{n}\right)^k \qquad (1.6)$$

where the $\gamma_{m,k}$ are certain fixed coefficients defined by a recursion [15]:

$$(1+2k/m)\,\gamma_{m,k} \;=\; (k-1-s/2)\,\gamma_{m,k-1} + \gamma_{m-1,k} \qquad (1.7)$$

for $m,k \geq 1$, this recursion being ignited by $\gamma_{0,k} := \delta_{0,k},\; \gamma_{m,0} := 1$.

3. **Recurrence relations:**

$$\begin{aligned} C_{0,0}(s,a) &:= a^{s/2}, \qquad (1.8)\\ C_{n,0}(s,0) &= B_n(s), \qquad (1.9)\\ asC_{m,0}(s-2,a) &= (s+m)C_{m,0}(s,a) - mC_{m-1,0}(s,a+1), \qquad (1.10)\\ (n+s)(n+s-1)B_n(s) &= s(n+s-2)B_n(s-2) + n(n-1)C_{n-2,0}(s,2). \qquad (1.11) \end{aligned}$$

2 State of the box-integral art

We have enumerated some important relations; presently we summarize known results in regard to specific dimensions.

To begin our summary, we note a recent definition of a class of closed forms. Following the treatment [6] we define a complex number to be *hyperclosed* if it belongs to the *ring of hyperclosure*. This ring is built as follows (see [6] for more details). Consider *generalized hypergeometric* evaluations

$$x = \sum_{n\geq 0} c_n z^n, \tag{2.1}$$

where z is algebraic, c_0 is rational, and $c_n = \frac{p(n)}{q(n)} c_{n-1}$ where p and q are polynomials with integer coefficients. Then the ring of hyperclosure is generated by all such evaluations x under $(+, \cdot)$. We say that any element of the ring is *hyperclosed*. So for example, π, $\pi + \log(2+\sqrt{7})$, $\mathrm{Li}_2(1/\sqrt{5}) + (\log 2)(\log 3)$ are all hyperclosed. Incidentally, lest one think the ring of hyperclosure is just too broad, note that said ring is *countable* [6].

Known results on box integrals include:

1. **Resolution of the Δ_n:** It was shown in the treatment [6, Theorem 7] that if all $B_m(s)$ *and* all residues Res_m, are known for $m \in [1, n]$, then $\Delta_n(s)$ is known. In that previous treatment, a form for the box residues is conjectured and some low-lying residues are given exactly. Happily, in the present work we shall establish a closed form for all Res_m with $m \in \mathbb{N}$. Thus, the problem of evaluating the Δ_n evaporates entirely, being replaced by the problem of evaluating the B_n.

2. **Hyperclosure in dimensions 1,2,3,4:** It is known that for any integer k, all $B_1(k)$, $B_2(k)$, $B_3(k)$, $B_4(k)$ are hyperclosed (exemplary closed forms appear in [6]). The same knowledge exists for $\Delta_n(k); n = 1, 2, 3, 4$ by the arguments of the previous item.

3. **Dimension 5:** It has been shown in previous work that for integer $k \neq -2, -4$, the box integral $B_5(k)$ is hyperclosed. The two exceptional k arguments have amounted to a "blockade"—with the obstacle amounting to a single, tough integral called $\mathcal{K}_5$, which integral we resolve below. This resolution of $\mathcal{K}_5$ established hyperclosure of all B_5(integer), and perforce all Δ_5(integer).

4. **Dimension 6:** Previous to our present treatment, not a single nontrivial B_6(integer) had been evaluated in closed form. (We say "nontrivial" here because, of course, $B_n(2h)$ for positive integer h is trivially rational.) But we are able to exhibit later in this paper the closed form for $B_6(-4)$, thus breaking the dimension-6 impasse. Moreover, using $B_6(-4)$ as a recursive pivot, we now know that B_6(even) are all hyperclosed. Virtually nothing is known, however, about B_6(odd).

5. **Very high dimension:** An algorithm has been discovered [15] that uses a series of the form (1.6) to resolve D digits of a box integral $B_n(s)$ in $O(n^2 D)$ operations, where the implied big-O constant depends only on s. D.H. Bailey has employed said algorithm to achieve a dimension-1-million box value to 100 good decimals, the value starting out

$$B_{1000000}(1) = 577.35021145457203997753408752036227457448125926146101942964\ldots$$

   It is evident that the art of *calculation* of the B_n is vastly ahead of the corresponding *symbolic* art.

3 Complete resolution of box residues

The general box-residue evaluation can be effected as follows. For n dimensions, the residue will be

$$\mathrm{Res}_n \;:=\; \lim_{\delta\to 0^+} \delta B_n(-n+\delta) = \lim_{\delta\to 0^+} \;\delta \int_{[0,1]^n} r^{-n+\delta}\; \mathcal{D}\vec{r}.$$

But the volume element is

$$\mathcal{D}\vec{r} = r^{n-1}\; dr\; \mathcal{D}\vec{\Omega},$$

where $\mathcal{D}\vec{\Omega}$ is a suitable definition of solid angle for n-space. Thus the r integral inside is

$$\int_0^{R(\vec{\Omega})} \frac{1}{r^{1-\delta}}\; dr,$$

where $R(\vec{\Omega})$ is the extent of radius from origin to the surface of the n-cube along direction $\vec{\Omega}$. The integral is

$$\frac{1}{\delta} R(\vec{\Omega})^{\delta},$$

which when multiplied by the outside factor δ can be taken to be 1. Thus, the residue comes down to being a piece (an n-tant) of the surface area of the unit n-sphere, namely

$$\mathrm{Res}_n = \frac{1}{2^{n-1}} \frac{\pi^{n/2}}{\Gamma(n/2)}. \tag{3.1}$$

This knowledge of all box residues leads, as we shall see, to new closed forms.

Another application of the general residue value is as follows. One of the present authors (O-Y C.) has conjectured that the pretty integral

$$I_n := \int_{[0,\pi/4]^n} \frac{d\theta_1\cdots d\theta_n}{(1+\sec^2\theta_1+\cdots+\sec^2\theta_n)^{1/2}}$$

is a rational multiple of π^n. This conjecture arose via numerical computation, then observation that the result matches sequence `A002457` in *Sloane's Online Encyclopedia* [25]. Here we prove the conjecture, giving as well the precise rational multiplier. First, it is a straightforward combinatorial result, via polar coordinates on each pair of integration variables in (1.3), that

$$C_{2n,0}(-2n-1,1) = \frac{2^n}{(2n-1)!!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} I_k\, \frac{\pi^{n-k}}{4^{n-k}}.$$

But by relation (1.5), the left-hand side is just $\mathrm{Res}_{2n+1}/(2n+1)$, which is given by (3.1). We now have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{I_k}{(\pi/4)^k} \;=\; \frac{1}{2n+1}.$$

The global solution to this recurrence system can be resolved by consequence of the *binomial transform* being an involution. We write

$$\left\{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{I_k}{(\pi/4)^k} \;=\; \frac{1}{2n+1}, \quad \forall n\in N\right\} \Leftrightarrow \left\{\left(\frac{\pi}{4}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2\,k+1} = I_n, \quad \forall n \in N\right\},$$

Summing the finite series on the right and simplifying we arrive that

$$\int_{[0,\pi/4]^k} \frac{d\theta_1\cdots d\theta_k}{(1+\sec^2\theta_1+\cdots+\sec^2\theta_k)^{1/2}} \;=\; \frac{k!^2}{(2k+1)!}\,\pi^k. \tag{3.2}$$

4 Some key integrals

To overcome the aforementioned box-integral blockade at 5 dimensions, and to move beyond into dimension-6 cases, we now establish closed forms for some key definite integrals. It is evident from the B, C relations in Section 1 that box integrals B_n depend intimately on $C_{m,0}$ integrals. We are thus interested in parameterized integrals $C_{m,0}(s, A)$ where the parameter A is a positive integer.[1] The most important instance for our present purposes can be developed via straightforward polar integration as

$$\begin{aligned} C_{4,0}(-4, A) &= -J(A+2) + \frac{1}{16}\pi^2 \log A - \frac{\pi}{2} G \\ &+ \frac{\pi}{4} \operatorname{Im}\left(\operatorname{Li}_2\left(\frac{i\left(A - 2\sqrt{A+1}+2\right)}{A}\right) + \operatorname{Li}_2\left(\frac{i\left(A + 2\sqrt{A+1}+2\right)}{A}\right)\right), \end{aligned} \tag{4.1}$$

see also (11.13). Here J is in turn a highly difficult integral we define by

$$J(t) \;:=\; \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1+x^2)(1+y^2)}\, dx\, dy.$$

We are interested in whether $J(t)$ is hyperclosed for algebraic t; accordingly, we begin with a new, fundamental result:

Lemma 4.1. *For any complex parameter t with $\operatorname{Re}(t) \geq 0$, we have*

$$J(t) = -\frac{\log 2}{2} \operatorname{Re}\left(\operatorname{Li}_2\left(\frac{2}{\sqrt{t-1}+1}\right) + \operatorname{Li}_2\left(-\frac{2}{\sqrt{t-1}-1}\right)\right) + R(t), \tag{4.2}$$

where

$$\begin{aligned} R(t) := \int_0^1 \log\left(\frac{1+x}{1-x}\right) &\left(\frac{x\log t - x\log(t+x^2+1)}{t-1-x^2}\right. \\ &\left. +(t-1)\frac{x\log(t+x^2+1) - x\log t - x\log(x^2+1)}{1-(t-1)x^2} + \frac{x\log(t+x^2+1)}{x^2+1} - \frac{x\log(x^2+1)}{x^2}\right) dx, \end{aligned} \tag{4.3}$$

with J interpreted at either of the special points $t = 1, 2$ via the limit of expression (4.2) for $t := q + \frac{1}{1-i\epsilon}$, $q = 0, 1$ and $\epsilon \to 0^+$.

Proof: We evaluate the integral $J(t)$ via a series of changes of variables. We begin by making a change into polar coordinates (r, θ), and then applying partial fractions, and make the change of variables $x^2 = \sec^2\theta - 1$ and $y = r^2$ to obtain

$$\begin{aligned} J(t) &= 2\int_0^{\pi/4} d\theta \int_0^{\sec\theta} r dr \frac{\log(t+r^2)}{(1+r^2\cos^2\theta)(1+r^2\sin^2\theta)} \\ &= \lim_{\delta\to 1} \int_0^{\delta} \frac{dx}{1-x^2} \int_0^{x^2+1} \log(t+y)\, dy \left(\frac{1}{x^2+1+y} - \frac{1}{x^{-2}+1+y}\right). \end{aligned}$$

The rest of the analysis (we omit many details here, to conserve space) employs the relations

$$\frac{d}{dx}\operatorname{Li}_2(-x) = -\frac{\log(1+x)}{x} \qquad \text{and} \qquad \frac{d}{dx}\operatorname{Li}_2(a/x) = \frac{\log(1-\frac{a}{x})}{x}, \tag{4.4}$$

[1] We choose to keep the second subscript 0 on $C_{m,0}$ just for consistency with previous treatments such as [6].

and the dilogarithm inversion relation

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2(-1/x) = -\frac{\pi^2}{6} - \frac{\log^2 x}{2}, \tag{4.5}$$

along with rather intricate integration-by-parts and partial fraction manipulations; all of this to obtain finally (4.2). **QED**

5 Overcoming the blockade at 5 dimensions

The integral known as $\mathcal{K}_5$, which appears in the relation

$$B_5(-4) \;=\; -5\,\mathcal{K}_5 - \frac{5}{2}\,\pi\,G + \frac{5}{4}\,\pi^2\,\log\left(1+\sqrt{2}\right) + \frac{5}{2}\,\pi\,\mathrm{Ti}_2\left(3-2\sqrt{2}\right),$$

was not resolved in [6]. Here Ti_2 is the generalized tangent of order two [20]. The literature definition can be cast as

$$\begin{aligned}\mathcal{K}_5 &:= \int_0^{\pi/4}\int_0^{\pi/4} \log(1+\sec^2 a+\sec^2 b)\; da\, db \\ &= \int_{[0,1]^2} \frac{\log(3+x^2+y^2)}{(1+x^2)(1+y^2)}\; dx\, dy = J(3).\end{aligned}$$

Thus if $J(3)$ is hyperclosed, then so is $B_5(-4)$. One can see clearly now why we have focused on the J integral.

With Lemma 4.1 we have reduced the problem of evaluating $J(t)$, and perforce $\mathcal{K}_5 = J(3)$, to that of evaluating $R(t)$. It will turn out that a necessary ingredient in our quest to pass the 5-dimensional blockade will be the invocation of *trilogarithms*, which entities had not yet appeared in any previous closed forms for box integrals. To this end, we invoke parameterized definite integrals, namely the pair

$$F^{\pm}(c) \;:=\; \int_0^1 \frac{\log(1\pm x)}{x+c}\; dx, \tag{5.1}$$

and the pair

$$G^{\pm}(b,c) \;:=\; \int_0^1 \frac{\log(1\pm x)\log(x+b)}{x+c}\; dx. \tag{5.2}$$

The closed forms for these four integrals are rather stultifying; we have relegated the closed-form displays to our Appendix I. The salient point being, we shall find that the remaining integral $R(t)$ can be cast as a superposition of F and G forms.

Theorem 5.1. *$J(t)$ is hyperclosed for algebraic t with* $\mathrm{Re}(t) \geq 0$.

(An explicit closed form is indicated in the proof following, although as before the special points $t = 0, 1$ need be handled as limiting values (see Appendix II).)

Proof: We write $R(t)$ as four definite integrals

$$R(t) = Y_1 + Y_2 + Y_3 + Y_4,$$

with the Y_n ordered as in the display of Lemma 4.1. To clarify, we have for example

$$Y_3 \ := \ \int_0^1 \log\left(\frac{1+x}{1-x}\right) \frac{x\log(t+x^2+1)}{x^2+1}\, dx.$$

Taking this exemplary case, we have—upon log-expansion and partial-fractions—the following:

$$Y_3 = \int_0^1 (\log(1+x) - \log(1-x)) \left(\log\left(x+i\sqrt{t+1}\right) + \log\left(x-i\sqrt{t+1}\right)\right) \frac{1}{2}\left(\frac{1}{x+i} + \frac{1}{x-i}\right)\, dx.$$

But this means Y_3 is a superposition of G^+, G^- evaluations—as is the case for each of the Y_n below if we also include F functions as needed:

$$Y_1 = \frac{1}{2}\left(\sum G^+\left(\pm i\sqrt{t+1}, \pm\sqrt{t-1}\right) - \sum G^-\left(\pm i\sqrt{t+1}, \pm\sqrt{t-1}\right)\right)$$
$$-\frac{\log t}{2}\left(\sum F^+\left(\pm\sqrt{t-1}\right) - \sum F^-\left(\pm\sqrt{t-1}\right)\right),$$

$$Y_2 = -\frac{1}{2}\left(\sum G^+\left(\pm i\sqrt{t+1}, \pm 1/\sqrt{t-1}\right) - \sum G^-\left(\pm i\sqrt{t+1}, \pm 1/\sqrt{t-1}\right)\right)$$
$$+\frac{1}{2}\left(\sum G^+\left(\pm i, \pm 1/\sqrt{t-1}\right) - \sum G^-\left(\pm i, \pm 1/\sqrt{t-1}\right)\right)$$
$$+\frac{\log t}{2}\left(\sum F^+\left(\pm 1/\sqrt{t-1}\right) - \sum F^-\left(\pm 1/\sqrt{t-1}\right)\right),$$

$$Y_3 = \frac{1}{2}\left(\sum G^+\left(\pm i\sqrt{t+1}, \pm i\right) - \sum G^-\left(\pm i\sqrt{t+1}, \pm i\right)\right),$$

$$Y_4 = -G\pi + \frac{7}{4}\zeta(3).$$

with each sum for an F-function performed over sign choices $\pm$, and for a G-function over all four possible sign choices $\pm\pm$. (One might object that Y_2 in particular involves integration over a branch singularity when $t > 2$, but in fact the integrand for Y_2 can be seen to be finite and differentiable; that is, any branch-cut effects are cancelled in the given superposition for Y_2.)

We therefore have a closed form based on the above closed forms for the Y_n, namely

$$J(t) = -\frac{\log 2}{2} \operatorname{Re}\left(\operatorname{Li}_2\left(\frac{2}{\sqrt{t-1}+1}\right) + \operatorname{Li}_2\left(-\frac{2}{\sqrt{t-1}-1}\right)\right)$$
$$+ \left(R(t) \ = \ Y_1 + Y_2 + Y_3 + Y_4\right).$$

For algebraic t this is a hyperclosed representation of $J(t)$, being as our Appendix forms for $F^{\pm}, G^{\pm}$ are accordingly hyperclosed for such t. **QED**

Because $\mathcal{K}_5 = J(3)$ has been the sole obstacle to 5-dimensional hyperclosure, we now have

Corollary 5.2. *The integral $\mathcal{K}_5$ is hyperclosed, and therefore all box integrals B_5(integer), perforce Δ_5(integer) are hyperclosed.*

The J integral at integer arguments other than $t = 3$ are interesting in their own right. Our Appendix II discusses closed-form developments for these cases.

5.1 A generalization of J

We conclude this portion of the paper by placing $J(A)$ in a more general context. Recall that

$$\begin{aligned} B_{2n+1}(-s) &:= \int_{[0,1]^{2n+1}} (x_1^2+\cdots+x_{2n+1}^2)^{-s/2}\, dx_1\cdots dx_{2n+1} = \frac{2n+1}{2n+1+s} C_{2n,0}(-s,1) \\ &= \frac{2n+1}{2n+1+s}\int_{[0,1]^{2n}} (1+x_1^2+\cdots+x_{2n}^2)^{-s/2}\, dx_1\cdots dx_{2n}. \end{aligned}$$

For general $C_{2n,0}$, we may change pairs of variables into polar coordinates to obtain

$$\begin{aligned} C_{2n,0}(-s,A) &:= \int_{[0,1]^{2n}} (A+x_1^2+x_2^2+\cdots+x_{2n}^2)^{-s/2}\, dx_1\cdots dx_{2n} \\ &= \frac{2^n}{(2-s)(4-s)\cdots(2n-s)} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} I_{n,k}(s,A)\frac{\pi^{n-k}}{4^{n-k}}, \end{aligned}$$

where

$$I_{n,k}(s,A) := \int_{[0,\pi/4]^k} \frac{d\theta_1\cdots d\theta_k}{(A+\sec^2\theta_1+\cdots+\sec^2\theta_k)^{s/2-n}}$$

provided that $s \neq 2,4,\ldots,2n$. The further change of variable $x_i^2 = \sec^2\theta_i - 1$ brings us back to the unit k-cube so that

$$I_{n,k}(s,A) = \int_{[0,1]^k} \frac{dx_1}{x_1^2+1}\cdots\frac{dx_k}{x_k^2+1}\frac{1}{(A+k+x_1^2+\cdots+x_k^2)^{s/2-n}}.$$

In the case $s=2n$, we find that

$$B_{2n+1}(-2n) = \frac{(-1)^{n-1}}{(2n-1)(n-1)!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{\pi^{n-k}}{4^{n-k}}\, J_k(k+1) \tag{5.3}$$

where $J_0(A) := 1$ and for $k \geq 1$ we have

$$\begin{aligned} J_k(A) &:= \int_{[0,1]^k} \frac{dx_1}{x_1^2+1}\cdots\frac{dx_k}{x_k^2+1}\log(A+x_1^2+\cdots+x_k^2) \qquad (5.4) \\ &= \int_{[0,\pi/4]^k} \log\left(A+\tan^2\theta_1+\cdots+\tan^2\theta_k\right)\, d\theta_1\cdots d\theta_k. \end{aligned}$$

Moreover, $J_{k+1}(A) = \int_0^{\pi/4} J_k\left(A+\tan^2 u\right)\, du$.

To illustrate,

$$J_2(A) = J(A) = \int_0^1\int_0^1 \frac{\log(A+x^2+y^2)}{(1+x^2)(1+y^2)}\, dxdy,$$

and $J_2(3) = J(3) = \mathcal{K}_5$, while a less difficult version of Lemma 4.1 shows

$$\begin{aligned} J_1(A) &= \int_0^{\pi/4} \log\left(A+\tan^2 t\right)\, dt = \frac{\pi}{4}\log(A-1) + \left(\theta-\frac{\pi}{4}\right)\log\left(\frac{\sqrt{A}-1}{\sqrt{A}+1}\right) \\ &+ \frac{1}{2}\operatorname{Cl}_2(\theta) - \frac{1}{2}\operatorname{Cl}_2(\theta-\pi) - G. \end{aligned}$$

where $\theta := \arctan\left(\frac{A-1}{2\sqrt{A}}\right) = 2\arctan\left(\frac{\sqrt{A}-1}{\sqrt{A}+1}\right)$.

Hence (5.3) and (5.4) show that $B_7(-6)$ is hyperclosed iff $J_3(4)$ is; and suggest that the evaluation of $B_{2n+1}(-2n)$ can be achieved in terms of Cl_k for $2 \leq k \leq n+1$.

6 The concept of "expression entropy"

Though box integrals B_5(integer) are now known to be hyperclosed, it is stultifying that the closed form for something like $B_5(-6)$ occupies one line of typical typesetting, yet our initial closed form for $J(3)$ (essentially $B_5(-4)$) from Theorem 5.1 had on the order of 10^5 characters. Furthermore, different symbolic languages[2] would use differing character counts to express $J(3)$.

We are motivated thus to introduce the notion of *expression entropy*—this will be the number of binary bits inherent in an expression. And, we found a very simple practical means of measuring such entropy. Namely: *run an expression as a text file, through an established entropy compressor.*

For example, our initial $J(3)$ expression, when entropy-compressed, reduced down to about $5 \cdot 10^4$ bits. Note that an initial 10^5 text characters is on the order of 10^6 bits, so this is significant reduction. One way to think of how expression entropy works is to consider that a good entropy compressor will essentially not care whether dilogarithms are written out Li2$(\dots)$ or PolyLog$[2, \dots]$, since the compressor is typically looking for common strings.

The reason why the expression-entropy concept is useful can be inferred from our Appendix commentary—note that $J(1), J(2), J(3)$, and $J(4)$ each start out as many pages of text, and even at current reductions $J(4)$ still barely fits on a single page. Yet $J(2)$ has reduced to a single line. The current work makes it abundantly clear that much work remains to be done regarding effective closed forms. Both *Maple* and *Mathematica* were able to rapidly confirm symbolic closed forms numerically but unable to produce symbolically in every case humanly convenient expressions.

7 6-dimensional box integrals

7.1 B_6(even)

For general dimension n, the recurrence (1.11) with $s = -n + 2$ yields

$$2B_n(-n+2) = (-n+2)\mathrm{Res}_n + n(n-1)C_{n-2,0}(2-n, 2).$$

Importantly, this reduces this box-integral B_n to an integral of dimension $(n-2)$. For example, the previously unresolved $B_6(-4)$ can now be written using the known residue Res_6 as

$$B_6(-4) = -\frac{\pi^3}{32} + 15C_{4,0}(-4, 2). \tag{7.1}$$

From the knowledge (4.1) and Theorem 5.1, it follows that $B_6(-4)$ is hyperclosed.

But we can carry this 6-dimensional effort yet further. The recursion relations such as (1.10, 1.11) give us the two key recursions

$$2sC_{4,0}(s-2, 2) \;=\; (s+4)C_{4,0}(s, 2) - 4C_{3,0}(s, 3), \tag{7.2}$$

and

$$(s+6)(s+5)B_6(s) \;=\; s(s+4)B_6(s-2) + 30C_{4,0}(s, 2). \tag{7.3}$$

[2]In our case, *Maple* and *Mathematica* were used extensively throughout our efforts at expression reduction.

These recursions contain enough magic to resolve *all* of the box integrals $B_6(\text{even})$. Indeed, pivoting on our knowledge of $B_6(-4)$ as given above, (7.2) leads us to

$$B_6(-2) \;=\; -\frac{2}{3}B_6(-4) - \frac{\pi^3}{96} + 5C_{3,0}(-2,3).$$

Happily, it turns out that the techniques used in our main Lemma 4.1 result in a closed form for $C_{3,0}(-2,A)$ as displayed—and refined—in our Appendix 11.2. So now we know $B_6(-2,-4)$ are both hyperclosed.

For the other direction away from $B_6(-4)$, we observe from (1.11) with $s \to -6$ that

$$B_6(-8) \;=\; -\frac{\pi^3}{768} - \frac{5}{2}C_{4,0}(-6,2).$$

It turns out we do not need to labor on the development of the C evaluation here, because it is elementary from the very definition (1.3) that

$$C_{m,0}(s,A) \;=\; \frac{2}{s+2}\frac{\partial}{\partial A}C_{m,0}(s+2,A), \tag{7.4}$$

so that

$$B_6(-8) \;=\; -\frac{\pi^3}{768} + \frac{5}{4}\frac{\partial}{\partial A}C_{4,0}(-4,A)|_{A=2},$$

thus $B_6(-8)$ is hyperclosed. Indeed $C_{4,0}(-4,A)$ is hyperclosed by (4.1), while the derivative of a hyperclosed expression with respect to one algebraic parameter is itself hyperclosed, via the definition of generalized hypergeometric functions and of the ring of hyperclosure.

Further $B_6(-10,-12,-14,\cdots)$, can now be resolved recursively using (7.3) in tandem with (7.4) with $m=4$. This all leads to

Theorem 7.1. *Every* $B_6(\text{even})$ *is hyperclosed.*

7.2 $B_6(\text{odd})$

In striking contrast to Theorem 7.1, we do not know a single $B_6(\text{odd})$ in closed form. The best we can do at the current juncture is to obtain forms with yet new kinds of dangling integrals, e.g.

$$B_6(-7) \;=\; -\frac{\pi^2}{\sqrt{8}} + \sqrt{8}\,\pi\arctan\left(\frac{1}{\sqrt{2}}\right) - 8\int_0^{\pi/4}\int_0^{\pi/4}\frac{dt\;du}{\sqrt{2+\sec^2 t+\sec^2 u}}.$$

For general parameter A one has

$$\int_0^{\pi/4}\int_0^{\pi/4}\frac{dt\;du}{\sqrt{A+\sec^2 t+\sec^2 u}} \;=\; \int_0^1 \frac{\arctan\left(\frac{\sqrt{A+1+y^2}}{\sqrt{A+3+y^2}}\right)}{(1+y^2)\sqrt{A+1+y^2}}\,dy.$$

Interestingly, we *do* know that this dangling integral for $A=1$ is equal to $\pi^2/30$, on the basis of (3.2); yet, we do not yet know the $A=2$ evaluation which would yield $B_6(-7)$. It would surprise us somewhat if resolutions of such elusive box integrals involve more than trilogarithmic terms, based on heuristics we shall explore in Section 8.

Somewhat more can be said about $B_6(\text{odd})$. By combining recursions (1.10, 1.11) one can deduce

$$360\,C_{2,0}(s,4) = \;-\; 6s^2(s-2)(s-4)B_6(s-6) + (s-2)s(s+2)(11s+18)B_6(s-4) \tag{7.5}$$

$$- \quad (3+s)(4+s)\left\{2s(3s+8)B_6(s-2)-(s+5)(s+6)B_6(s)\right\}.$$

We next observe that $C_{2,0}(s,A)$ is hyperclosed for odd integers s. We write

$$C_{2,0}(s,A) = \frac{2}{s+2}\, M(s+2,A) - \frac{\pi}{2}\, \frac{A^{s/2+1}}{s+2},$$

where $M(s,A) := \int_0^{\pi/4} \left(A+1+\tan^2\theta\right)^{s/2} d\theta$. We have a recursion

$$M(s+2,A) - A\, M(s,A) = (A+1)^{s/2}\, {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -\frac{1}{A+1}\right),$$

so that the hyperclosure of all $C_{2,0}(\mathrm{odd},4)$ is ignited by a single evaluation, say $C_{2,0}(-1,4) = 4\arctan\left(\sqrt{\frac{2}{3}}\right) - \pi + \log 5 - 2\log\left(\sqrt{6}-1\right)$. One might guess that because the $C_{2,0}(s,4)$ on the left of recursion (7.5) is thus hyperclosed, the $B_6(\mathrm{odd})$ should be determined by B_6 at any three consecutive arguments. But this is not so: Sometimes the coefficients in (7.5) vanish. By looking closely at cases such as $s = -1, -3, -5, ...$ one can establish

Theorem 7.2. *If $B_6(-7)$ is hyperclosed then so are all of $B_6(-9,-11,-13,\ldots)$. If in addition $B_6(-3,-5)$ are hyperclosed, then so are all $B_6(\mathrm{odd})$.*

Therefore, resolution of all $B_6(\mathrm{integer})$ would follow from hyperclosure of the three entities $B_6(-3,-5,-7)$; we remind ourselves we do not know a closed form for any one of these three.

8 Theory of hyperdegree

Based on previous box-integral research, together with the present treatment, an interesting pattern emerges in regard to the "polylogarithmic degree" of various evaluations. We hereby define *hyperdegree* as a certain measure on the ring of hyperclosure, and mean this to be more general than polylogarithmic degree. We shall speak heuristically in what follows, because it is very hard to produce rigorous results in this area. One might say that in looking for hyperdegree patterns we are observing "evident hyperdegree" without proof.[3]

To define hyperdegree $H(X)$ of a ring element X, we start with ring-generator evaluations (2.1) and define $H(x)$ as the minimal degree of the denominator polynomial $q(n)$, over *all* hypergeometric expansions of x. For example (here z denotes an algebraic number)

$$H(z) = 0,$$

$$H(\log z) \leq 1,$$

$$H\left(\mathrm{Li}_n(z)\right) \leq n.$$

Note that we use '$\leq$' sometimes because there can be exceptions, e.g. $\log 1 = 0$, and also because, again, we cannot always prove exact hyperdegrees. Some isolated cases can be proven, such as

$$H(\pi) = 1,$$

whose proof is an instructive exercise. Generally speaking, when we obtain a closed form, we have an upper bund on the hyperdegree H.

[3]For example, is the hyperdegree of $\zeta(5)$ equal to 5? Not if $\zeta(5)$ is rational!

n	$H(B_n(\text{even})) \ \le$	$H(B_n(\text{odd})) \ \le$
2	1	1
3	2	1
4	2	2
5	3	2
6	3	3 ?
7	4 ?	3 ?
$n > 1$	$\lceil n/2 \rceil$?	$\lfloor n/2 \rfloor$?

Table 1: Known bounds on hyperdegrees for box integrals. All hyperdegree entries without '?' are rigorous upper bounds (since we know the relevant closed forms). The immediate conjectures are that $H(B_6(\text{odd})) \le 3$, that the hyperdegrees for B_7 are $4, 3$, and that for general dimension $n > 1$ we have the indicated floor, ceiling bounds.

Next we define the hyperdegree of a ring element X, by using the symbology

$$X = \sum_j \left(\prod_k x_{j,k} \right),$$

where the $x_{j,k}$ are an array—not necessarily rectangular—of generator evaluations, and both sum and product here are finite. We define

$$H(X) \ := \ \min_{()} \left(\max_j \left(\sum_k H(x_{j,k}) \right) \right)$$

The meaning of $\min_{()}$ means to take the minimum of *all* ring representations $x_{j,k}$ for X. (Such is necessary to guard against resonances such as $H(\pi^2 - \pi^2) = 0$, yet each component π^2 has hyperdegree 2.) The point of the rather erudite notation can be intuitively expressed thus: The hyperdegree of a ring element is the largest hyperdegree of an isolated product string, which is in turn (at most) the sum of the hyperdegrees of the $x_{j,k}$ members of said string. With such notions—admittedly not entirely rigorous—of hyperdegree, we have examples such as

$$H\left(1 + \log 7 - \log 2 \log^2 3 - \pi^3 + \mathrm{Li}_3\left(\sqrt{5} - 2\right)\right) \ \le \ 3,$$

and this H probably *is* 3. A specific and relevant box-integral example is

$$H\left(B_3(-2)\right) = H\left(-3G + \frac{3}{2}\pi \log(1 + \sqrt{2}) + 3 \ \mathrm{Ti}_2(3 - 2\sqrt{2})\right) \ \le 2.$$

The implications of hyperdegree theory for box integrals can be gleaned from Table 1, which table uses results from [6] together with our new results on B_5, B_6. Note that the development of section 5.1 also adds substance to this conjecture.

9 Conclusion

Salient open issues include:

- Proof that some or all $B_6(\text{odd})$ are hyper-closed.
- Evaluation of some $B_n(s)$ for $n \geq 7$; ideally of an infinite family such as $B_n(-n-1)$ or $B_{n+1}(-n)$ for $n \geq 6$.
- A better understanding of the structure of such evaluations.

Each of these is relevant also to other physically motivated classes of integrals [7].

Finally, one of the largest challenges for such computer-assisted analysis is to automate the process described in Appendices I and II so that results like (11.16) can be obtained by the computer with at most limited human agency. The appendices provide invaluable test data for such projects—and one such project is currently being undertaken.

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10 Appendix I: The functions $F^\pm$ and $G^\pm$

The $F^\pm$ integrals resolve as

$$F^+(c) = -\mathrm{Li}_2\left(\frac{1}{1-c}\right) + \mathrm{Li}_2\left(-\frac{2}{c-1}\right) + \log 2 \log\left(\frac{c+1}{c-1}\right),$$

$$F^-(c) = -\mathrm{Li}_2\left(\frac{1}{c+1}\right).$$

It is already of interest that these two $\pm$ forms seem to differ in complexity. In fact, trying an integrand factor $\log(1+sx)$ and expecting to take $s = \pm 1$ in a general integral involves a nontrivial limit for the $s = -1$ case. All of this difficulty can be traced to the well known branch-cut peculiarity of the dilogarithm $\mathrm{Li}_2(z)$ on $z \in (+1, +\infty)$.

The $G^\pm$ integrals are far more intricate; and again, it is best to handle the $\pm$ cases separately. The following exact analytic forms can be gleaned from fundamental trilogarithmic formulae such as [20, Eq 8.111]. (It is again interesting that G^- is evidently less complex than G^+, although it is unclear what further symbolic reductions might be possible for either $G^\pm$.)

$$\begin{aligned}
G^-(b,c) = {} & \mathrm{Li}_3\left(\frac{b}{b-c}\right) - \mathrm{Li}_3\left(\frac{b+1}{b-c}\right) - \mathrm{Li}_3\left(\frac{b(c+1)}{b-c}\right) - \mathrm{Li}_2(-b)\log\left(\frac{b(c+1)}{b-c}\right) \\
& - \mathrm{Li}_2\left(\frac{b}{b-c}\right)\log\left(\frac{b(c+1)}{b-c}\right) + \mathrm{Li}_2\left(\frac{b+1}{b-c}\right)\log\left(\frac{(b+1)(c+1)}{b-c}\right) - \log b\ \mathrm{Li}_2\left(\frac{1}{c+1}\right) \\
& + \mathrm{Li}_2\left(\frac{1}{c+1}\right)\log\left(\frac{b(c+1)}{b-c}\right) + \mathrm{Li}_2\left(\frac{b(c+1)}{b-c}\right)\log\left(\frac{b(c+1)}{b-c}\right) \\
& - \frac{1}{6}\log^3\left(-\frac{(b+1)(c+1)}{b-c}\right) - \frac{1}{2}\log^2(b+1)\log\left(\frac{(b+1)(c+1)}{b-c}\right) \\
& - \frac{1}{2}\log(b+1)\log^2\left(\frac{b(c+1)}{b-c}\right) + \frac{1}{2}\log(b+1)\log^2\left(\frac{(b+1)(c+1)}{b-c}\right) \\
& - \frac{1}{2}\log c\log^2\left(\frac{b(c+1)}{b-c}\right) + \frac{1}{2}\log\left(-\frac{(b+1)c}{b-c}\right)\log^2\left(\frac{b(c+1)}{b-c}\right) \\
& + \frac{1}{2}\log(c+1)\log^2\left(\frac{b(c+1)}{b-c}\right) - \frac{1}{2}\log\left(-\frac{(b+1)(c+1)}{b-c}\right)\log^2\left(\frac{(b+1)(c+1)}{b-c}\right) \\
& - \frac{1}{2}\log c\log^2\left(\frac{b}{b-c}\right) + \frac{1}{2}\log^2\left(\frac{b}{b-c}\right)\log\left(-\frac{c}{b-c}\right) + \frac{1}{2}\log(c+1)\log^2\left(\frac{b}{b-c}\right) \\
& + \frac{1}{2}\log^2\left(-\frac{(b+1)(c+1)}{b-c}\right)\log\left(\frac{(b+1)(c+1)}{b-c}\right) - \frac{1}{2}\log^2\left(\frac{b+1}{b-c}\right)\log\left(\frac{c+1}{c-b}\right) \\
& - \frac{1}{6}\pi^2\log\left(-\frac{c+1}{b-c}\right) - \log\left(\frac{b}{b-c}\right)\log\left(-\frac{c}{b-c}\right)\log\left(\frac{b(c+1)}{b-c}\right) \\
& + \log\left(\frac{c}{c+1}\right)\log\left(\frac{b}{b-c}\right)\log\left(\frac{b(c+1)}{b-c}\right) \\
& + \log\left(\frac{b+1}{b-c}\right)\log\left(\frac{(b+1)(c+1)}{b-c}\right)\log\left(\frac{c+1}{c-b}\right) + \mathrm{Li}_3(-b) + \frac{1}{6}\log^3(b+1) \\
& + \mathrm{Li}_3\left(\frac{1}{c+1}\right).
\end{aligned}$$

$$
\begin{aligned}
G^{+}(b,c) \;=\;& \frac{1}{2}\log\left(-\frac{c}{b-c}\right)\log^2\left(\frac{b}{b-c}\right)-\frac{1}{2}\log\left(\frac{c}{c-1}\right)\log^2\left(\frac{b}{b-c}\right)\\
&-\log\left(-\frac{b(c-1)}{b-c}\right)\log\left(-\frac{c}{b-c}\right)\log\left(\frac{b}{b-c}\right)\\
&+\log\left(-\frac{b(c-1)}{b-c}\right)\log\left(\frac{c}{c-1}\right)\log\left(\frac{b}{b-c}\right)-\frac{1}{2}\log(1-b)\log^2\left(-\frac{b(c-1)}{b-c}\right)\\
&+\frac{1}{2}\log^2 2\log(1-b)+\frac{1}{2}\log(1-b)\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(-\frac{(b+1)(c-1)}{4(b-c)}\right)\\
&+\frac{1}{2}\log^2\left(-\frac{b(c-1)}{b-c}\right)\log\left(\frac{(b-1)c}{b-c}\right)-\frac{1}{2}\log^2\left(-\frac{b(c-1)}{b-c}\right)\log\left(\frac{c}{c-1}\right)\\
&-\frac{1}{2}\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(-\frac{(b+1)(c-1)}{4(b-c)}\right)\log\left(\frac{(b-1)(c+1)}{b-c}\right)\\
&-\frac{1}{2}\log^2 2\log\left(\frac{(b-1)(c+1)}{b-c}\right)+\frac{1}{2}\log^2\left(\frac{b+1}{b-c}\right)\log\left(\frac{c+1}{c-1}\right)\\
&+\frac{1}{2}\log^2\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(\frac{c+1}{c-1}\right)+\log 2\log\left(1+\frac{1}{b}\right)\log\left(\frac{c+1}{c-1}\right)\\
&+\log 2\log b\log\left(\frac{c+1}{c-1}\right)-\log\left(\frac{b+1}{b-c}\right)\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(\frac{c+1}{c-1}\right)\\
&-\log 2\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(\frac{c+1}{c-1}\right)+\frac{1}{2}\log^2 2\log\left(\frac{c+1}{c-1}\right)\\
&-\frac{1}{2}\log^2\left(\frac{b+1}{b-c}\right)\log\left(\frac{c+1}{c-b}\right)+\log\left(\frac{b+1}{b-c}\right)\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\log\left(\frac{c+1}{c-b}\right)\\
&-\log\left(-\frac{b(c-1)}{b-c}\right)\mathrm{Li}_2(b)+\log\left(-\frac{(b+1)(c-1)}{2(b-c)}\right)\mathrm{Li}_2\left(\frac{b+1}{2}\right)-\log b\;\mathrm{Li}_2\left(\frac{1}{1-c}\right)\\
&+\log\left(-\frac{b(c-1)}{b-c}\right)\mathrm{Li}_2\left(\frac{1}{1-c}\right)-\log\left(-\frac{b(c-1)}{b-c}\right)\mathrm{Li}_2\left(\frac{b}{b-c}\right)\\
&+\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\mathrm{Li}_2\left(\frac{b+1}{b-c}\right)+\log\left(2+\frac{2}{b}\right)\mathrm{Li}_2\left(-\frac{2}{c-1}\right)\\
&+\log b\;\mathrm{Li}_2\left(-\frac{2}{c-1}\right)-\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\mathrm{Li}_2\left(-\frac{2}{c-1}\right)\\
&-\log\left(-\frac{(b+1)(c-1)}{b-c}\right)\mathrm{Li}_2\left(-\frac{(b+1)(c-1)}{2(b-c)}\right)+\log 2\;\mathrm{Li}_2\left(-\frac{(b+1)(c-1)}{2(b-c)}\right)\\
&+\log\left(-\frac{b(c-1)}{b-c}\right)\mathrm{Li}_2\left(\frac{b-bc}{b-c}\right)+\mathrm{Li}_3(b)-\mathrm{Li}_3\left(\frac{b+1}{2}\right)\\
&+\mathrm{Li}_3\left(\frac{1}{1-c}\right)+\mathrm{Li}_3\left(\frac{b}{b-c}\right)-\mathrm{Li}_3\left(\frac{b+1}{b-c}\right)-\mathrm{Li}_3\left(-\frac{2}{c-1}\right)\\
&+\mathrm{Li}_3\left(-\frac{(b+1)(c-1)}{2(b-c)}\right)-\mathrm{Li}_3\left(\frac{b-bc}{b-c}\right).
\end{aligned}
$$

11 Appendix II: Explicit forms for $J(n), 0 \le n \le 4$, and $C_{3,0}(-2, A)$

The easiest value of J is $J(0)$ which we may obtain directly from the original integral form. We write

$$\begin{aligned} J(0) &= 2\int_0^{\pi/4}\int_0^a \frac{\log\left(a^2+b^2\right)}{(1+a^2)(1+b^2)}\,db\,da \qquad (11.1)\\ &= -\int_0^1 \frac{\log^2\left(\frac{c^2+1}{2}\right)\,dc}{1-c^2} + \frac{\pi^2}{16}\log(2) + \frac{7}{4}\zeta(3) - \pi\,\mathrm{G}. \end{aligned}$$

Moreover,

$$\int_0^1 \frac{\log^2\left(\frac{c^2+1}{2}\right)\,dc}{1-c^2} = \sum_{k=1}^{\infty} 2^k\,\frac{\sum_{j=1}^{k-1} 1/j}{\binom{2\,k}{k}k^2} = -\pi\mathrm{G} + \frac{21}{8}\zeta(3),$$

so that

$$J(0) = \frac{\pi^2}{16}\log 2 - \frac{7}{8}\zeta(3). \qquad (11.2)$$

We can evaluate $J(1)$ by much the same techniques as given above for $J(3)$. We obtain—in the limit—a ten-thousand character expression which with care reduces to:

$$\begin{aligned} J(1) = {} & \frac{9}{8}\pi^2\log 2 - \frac{3}{2}\mathrm{G}\,\pi - \frac{1}{2}\log^2 2\log 3 + \log^3 2 - \frac{7}{16}\pi^2\log 3\\ & + \log^2(1+\sqrt{2})\log 2 - \log^2(1+\sqrt{2})\log 3 - 2\pi\arctan\sqrt{2}\log 2 + \pi\arctan\sqrt{2}\log 3\\ & + \frac{1}{2}\pi\ \mathrm{Im}\left\{\mathrm{Li}_2\left(2-i+(1-i)\sqrt{2}\right) + \mathrm{Li}_2\left(2-i-(1-i)\sqrt{2}\right)\right\}\\ & + \mathrm{Re}\left[\left(i\pi - 2i\arctan\sqrt{2} + 3\log 2 - \log 3\right)\mathrm{Li}_2\left(\frac{1}{2} - \frac{1}{4}i\sqrt{2}\right)\right.\\ & \quad + \left(2i\pi - 4i\arctan\sqrt{2} + 2\log 3 - 4\log 2\right)\mathrm{Li}_2\left(\frac{1}{2} - \frac{1}{2}i\sqrt{2}\right)\\ & \quad + \left(\frac{3}{4}i\pi - i\arctan\sqrt{2} + \frac{1}{2}\log 2 - \frac{1}{2}\log 3 - \log\left(1+\sqrt{2}\right)\mathrm{Li}_2\left(\frac{1}{2} - \frac{3}{2}i - i\sqrt{2}\right)\right)\\ & \quad + \left(-\frac{1}{4}i\pi + i\arctan\sqrt{2} + \frac{1}{2}\log 2 - \frac{1}{2}\log 3 + \log\left(1+\sqrt{2}\right)\right)\mathrm{Li}_2\left(\frac{1}{2} - \frac{3}{2}i + i\sqrt{2}\right)\\ & \quad \left. +2\,\mathrm{Li}_3\left(\frac{1}{2} - \frac{1}{4}i\sqrt{2}\right) - 4\,\mathrm{Li}_3\left(\frac{1}{2} - \frac{1}{2}i\sqrt{2}\right) + \mathrm{Li}_3\left(\frac{1}{2} - \frac{3}{2}i + i\sqrt{2}\right) + \mathrm{Li}_3\left(\frac{1}{2} - \frac{3}{2}i - i\sqrt{2}\right) - 2\,\mathrm{Li}_3\left(1-i\right)\right]. \end{aligned}$$

We refine this evaluation below. As we shall see the expression of the integral $\mathcal{K}_5 = J(3)$ is similar but significantly more complex; the original expression being roughly four times as long.

11.1 Generalized Clausen functions and relatives

To further resolve $J(1), J(2), J(3)$ and $J(4)$ one has to consider complex polylogarithms and generalized *Clausen functions*.

Let us denote $\mathrm{Cl}_2(r,\theta) := \mathrm{Im}\,\mathrm{Li}_2\left(re^{i\theta}\right)$, as a counterpart to

$$\mathrm{Li}_2(r,\theta) := \mathrm{Re}\,\mathrm{Li}_2\left(re^{i\theta}\right) = -\frac{1}{2}\int_0^r \frac{\log(1-2r\cos(\theta)+r^2)}{r}\,d\theta,$$

so that $\mathrm{Cl}_2(1,\theta) = \mathrm{Cl}_2(\theta) = \sum_{n\ge 1} \frac{\sin(n\theta)}{n^2}$. (We recall that Cl_2 is a non-elementary Fourier series as opposed to $\mathrm{Li}_2(1,\theta)$.) Then, see [20, A2.5.(1)], one obtains

$$\mathrm{Cl}_2(r,\theta) = \frac{1}{2}\,\mathrm{Cl}_2(2\omega) + \frac{1}{2}\,\mathrm{Cl}_2(2\theta) - \frac{1}{2}\,\mathrm{Cl}_2(2\omega+2\theta) + w\log(r), \tag{11.3}$$

where $\tan(\omega) = r\sin(\theta)/(1-r\cos(\theta))$.

Also for $r>0$ and integer n, $\mathrm{Li}_2(r,\theta) = \mathrm{Li}_2(r, 2n\pi \pm \theta)$ and $\mathrm{Cl}_2(r,\theta) = -\mathrm{Cl}_2(r,-\theta)$. Very usefully, for $0 \le \theta < 2\pi$

$$\mathrm{Li}_2\left(r,\theta\right) + \mathrm{Li}_2\left(1/r,\theta\right) = \frac{1}{2}\,\left(\pi-\theta\right)^2 - \frac{1}{2}\,\log^2\left(r\right) - \frac{\pi^2}{6}. \tag{11.4}$$

We record the following useful Fourier series reductions:

$$\begin{aligned}
\mathrm{Li}_2\left(\tan\frac{\theta}{2},\theta\right) &= \frac{\theta^2}{4} + \frac{1}{2}\,\mathrm{Li}_2\left(\tan^2\left(\frac{\theta}{2}\right)\right) - \frac{1}{4}\,\mathrm{Li}_2\left(\tan^2\left(\frac{\theta}{2}\right)\right) &(11.5)\\
\mathrm{Li}_2\left(2\cos\theta,\theta\right) &= \left(\frac{\pi}{2}-\theta\right)^2\\
\mathrm{Li}_2\left(\cos\theta,\theta\right) &= \frac{1}{4}\,\mathrm{Li}_2\left(\cos^2\theta\right) + \frac{1}{2}\,\left(\frac{\pi}{2}-\theta\right)^2\\
\mathrm{Li}_2\left(\sec\theta,\theta\right) &= \frac{5}{24}\,\pi^2 - \frac{1}{4}\,\mathrm{Li}_2\left(\cos^2\theta\right) - \frac{1}{2}\,\log^2\left(\cos\theta\right) - \frac{\pi}{2}\,\theta\\
\mathrm{Li}_2\left(\frac{\sec\theta}{2},\theta\right) &= \frac{\pi^2}{12} - \frac{1}{2}\,\log^2\left(2\,\cos\theta\right) - \frac{\theta^2}{2}.
\end{aligned}$$

In particular, using the final formula we obtain

$$\mathrm{Li}_2\left(\frac{\sqrt{6}}{4}, \arctan\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{12} - \frac{1}{8}\,\log^2\frac{8}{3} - \frac{1}{2}\,\arctan^2\frac{1}{\sqrt{2}},$$

and

$$\mathrm{Li}_2\left(\frac{\sqrt{3}}{2}, \frac{\pi}{2} - \arctan\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{12} - \frac{1}{8}\,\log^2\frac{4}{3} - \frac{1}{2}\,\left(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{2}}\right)^2.$$

Likewise, see [20, A2.6], we notate $\mathrm{Li}_3(r,\theta) := \mathrm{Re}\,\mathrm{Li}_3\left(re^{i\theta}\right)$ so that in keeping with Lewin [20] $\mathrm{Cl}_3(\theta) := \mathrm{Li}_3(1,\theta)$. In particular [20, A2.5] gives various functional equations for $\mathrm{Li}_2(r,\theta)$ and [20, A2.6] gives various functional equations for $\mathrm{Li}_3(r,\theta)$. If we use the generalized tangent $T_3(\rho) = \sum_{n\ge 0}(-1)^n \frac{\rho^{2n+1}}{(2n+1)^3}$ then

$$\mathrm{Im}\,\mathrm{Li}_3\left(re^{i\theta}\right) = T_3(\rho) - T_3(\rho,\tan\theta)$$

where $\rho = r\sin\theta/(1-r\cos\theta)$ defines $T_3(\rho,\tan\theta)$. Also for $r>0$ we have $\mathrm{Li}_3(r,\theta) = -\mathrm{Li}_3(r,-\theta)$. Again, for $0\le\theta<2\pi$, we have

$$\mathrm{Li}_3\left(r,\theta\right) - \mathrm{Li}_3\left(1/r,\theta\right) = -\frac{1}{6}\log^3 r + \frac{3\left(\pi-\theta\right)^2 - \pi^2}{6}\log r. \tag{11.6}$$

For example,

$$\mathrm{Li}_3\left(\sqrt{2},\frac{\pi}{4}\right) = \mathrm{Li}_3\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right) + \frac{11\pi^2}{192}\,\log 2 - \frac{1}{48}\,\log^3 2 = \frac{35}{64}\,\zeta\left(3\right) + \frac{\pi^2}{32}\log 2.$$

We shall also exploit a consequence of the functional equation for the trilogarithm due to Landen [20, A2.6 (7)]:

$$\mathrm{Li}_3\left(x\right)+\mathrm{Li}_3\left(1-x\right)+\mathrm{Li}_3\left(\frac{-x}{1-x}\right)=\zeta(3)+\zeta(2)\log\left(1-x\right)+\frac{1}{2}\log^2\left(1-x\right)\log x+\frac{1}{6}\log^3\left(1-x\right). \tag{11.7}$$

This implies that for $0\le\omega\le 2\pi$ one has

$$\mathrm{Re}\,\mathrm{Li}_3\left(\frac{1}{2}+i\omega\right) = \frac{1}{2}\zeta\left(3\right)-\frac{1}{2}\mathrm{Cl}_3\left(\sigma\right)+-\frac{1}{48}\log^3\left(\frac{1+4\,\omega^2}{4}\right)+\frac{1}{4}\left(\frac{\pi^2}{6}-\arctan^2\left(2\,\omega\right)\right)\log\left(\frac{1+4\,\omega^2}{4}\right). \tag{11.8}$$

Here $\sigma := \arctan\left(\frac{4\,\omega}{4\,\omega^2-1}\right)$ is the principle value in $(-\pi,\pi]$.

The corresponding formula for the dilogarithm is

$$\mathrm{Re}\,\mathrm{Li}_2\left(\frac{1}{2}+i\omega\right)=\frac{\pi^2}{12}-\frac{1}{8}\log^2\left(\frac{1+4\omega^2}{4}\right)-\frac{1}{2}\arctan^2\left(2\,\omega\right). \tag{11.9}$$

11.2 A closed form for $C_{3,0}(-2,A)$

We now have the requisite tools to produce an explicit closed form for $C_{3,0}(-2,A)$ as required to establish hyperclosure of all B_6(even). We recall that

$$C_{3,0}(-2,A)=\int_{[0,1]^3}\frac{dxdydz}{A+x^2+y^2+z^2}$$

The formula that comes out of our intricate integration, using the techniques behind Lemma 4.1, is as below:

$$\begin{aligned}
2C_{3,0}(-2,A) &= \frac{3}{2}\log^2 2+\log 2\log(A-1)-\pi\sqrt{A}\arctan\frac{1}{\sqrt{A}} \qquad (11.10)\\
&+ \mathrm{Li}_2\left(-\frac{A-1}{2}\right)+\mathrm{Li}_2\left(-\frac{2}{A-1}\right)-\mathrm{Li}_2\left(-\frac{A-1}{4}\right)-\mathrm{Li}_2\left(-\frac{4}{A-1}\right) \qquad (a)\\
&+ 6\,\mathrm{Im}\left[\mathrm{Li}_2\left(\frac{1+i}{1+\sqrt{A+2}}\right)+\mathrm{Li}_2\left(\frac{-1+i}{-1+\sqrt{A+2}}\right)\right] \qquad (b)\\
&+ 2\sqrt{A}\,\mathrm{Re}\left[\mathrm{Li}_2\left(\frac{(1+\sqrt{A})(\sqrt{A}+i\sqrt{A+2})}{2(A+1)}\right)-\mathrm{Li}_2\left(-\frac{(-1+\sqrt{A})(\sqrt{A}+i\sqrt{A+2})}{2(A+1)}\right)\right.\\
&\quad+\mathrm{Li}_2\left(\frac{(1-\sqrt{A})(1-i\sqrt{A}\sqrt{A+2})}{(A+1)^2}\right)-\mathrm{Li}_2\left(\frac{(1+\sqrt{A})(1+i\sqrt{A}\sqrt{A+2})}{(A+1)^2}\right)\\
&\quad\left.+\mathrm{Li}_2\left(\frac{(1+\sqrt{A})(1+i\sqrt{A})}{A+1}\right)-\mathrm{Li}_2\left(\frac{(1-\sqrt{A})(1-i\sqrt{A})}{A+1}\right)\right] \qquad (c).
\end{aligned}$$

We next rewrite the dilogarithmic terms of $\mathrm{Li}_2(r,\theta)$ and $\mathrm{Cl}_2(r,\theta)$ values. For (a) we appeal twice to (11.4) with $\theta=\pi$. We discover that

$$\mathrm{Li}_2\left(-\frac{A-1}{2}\right)+\mathrm{Li}_2\left(-\frac{2}{A-1}\right)-\mathrm{Li}_2\left(-\frac{A-1}{4}\right)-\mathrm{Li}_2\left(-\frac{4}{A-1}\right)=\frac{3}{2}\log^2 2-\log 2\log\left(A-1\right).$$

We write (b) as

$$6\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{1+i}{1+\sqrt{A+2}}\right) \quad + \quad 6\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{-1+i}{-1+\sqrt{A+2}}\right) = 6\,\mathrm{Cl}_2\left(\frac{\sqrt{2}}{1+\sqrt{A+2}},\frac{\pi}{4}\right) - 6\,\mathrm{Cl}_2\left(\frac{\sqrt{2}}{-1+\sqrt{A+2}},\frac{3\pi}{4}\right)$$
$$= \quad 6\,G - 3\,\mathrm{Cl}_2\left(\frac{\pi}{2}+\kappa\right) - 3\,\mathrm{Cl}_2\left(\frac{\pi}{2}-\kappa\right) + 3\kappa\,\log\left(\frac{\sqrt{A+2}+1}{\sqrt{A+2}-1}\right)$$

where $\kappa := 2\,\arctan\left(\frac{1}{\sqrt{A+2}}\right)$.

Similarly (c) is actually three pairs of $\mathrm{Li}_2(r,\theta)$ values with equal angles and twisted moduli. Let $\nu := \arctan\left(\sqrt{A}\right), \eta : \arctan\left(\sqrt{A(A+2)}\right), \omega := \arctan\left(\sqrt{(A+2)/A}\right)$; note that $2\omega + \eta = \pi$.

Then putting everything together for $A \geq 1$ we have:

$$\begin{aligned} C_{3,0}(-2,A) \quad &= \quad \frac{1}{2}\log^2 2 - \frac{\pi^2}{8}\sqrt{A} - \frac{\pi}{2}\kappa\sqrt{A} + 6\,G - 3\,\mathrm{Cl}_2\left(\frac{\pi}{2}+\kappa\right) - 3\,\mathrm{Cl}_2\left(\frac{\pi}{2}-\kappa\right) + 3\kappa\,\log\left(\frac{\sqrt{A+2}+1}{\sqrt{A+2}-1}\right) \\ &+ \quad \sqrt{A}\left\{\mathrm{Li}_2\left(\frac{\sqrt{2}\left(\sqrt{A}+1\right)}{(A+1)^{3/2}},\omega\right) - \mathrm{Li}_2\left(\frac{\sqrt{2}\left(\sqrt{A}-1\right)}{(A+1)^{3/2}},\omega\right) + \mathrm{Li}_2\left(\frac{\sqrt{A}-1}{A+1},\pi-\eta\right)\right. \\ &\left. - \mathrm{Li}_2\left(\frac{\sqrt{A}+1}{A+1},\eta\right) + \mathrm{Li}_2\left(\frac{\sqrt{A}+1}{\sqrt{A+1}},\pi-\nu\right) - \mathrm{Li}_2\left(\frac{\sqrt{A}-1}{\sqrt{A+1}},\nu\right)\right\}. \end{aligned} \tag{11.11}$$

This form is especially attractive when $A = 1$, so that $\kappa = \omega = \eta = \pi/3$ and $\nu = \pi/4$. On noting that

$$4G \quad = \quad 3\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) + 3\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right) \tag{11.12}$$
$$\mathrm{Li}_2\left(\sqrt{2},\frac{\pi}{4}\right) \quad = \quad \frac{\pi^2}{16}, \quad \mathrm{Li}_2\left(1,\frac{\pi}{4}\right) = \frac{11\pi^2}{192}, \quad \mathrm{Li}_2\left(1,\frac{3\pi}{4}\right) = \frac{13\pi^2}{192}$$

it reduces to $2\,C_{3,0}(-2,1) = B_4(-2) = \pi\log(2+\sqrt{3}) - 2G + \pi^2/8$ as obtained in [6].

We can similarly rework (4.1) as below:

$$C_{4,0}(-4,A) = \frac{\pi^2}{16}\log A - \frac{\pi}{2}G \;+ \frac{\pi}{4}\mathrm{Cl}_2\left(\frac{\sqrt{A+1}+1}{\sqrt{A}-1},\frac{\pi}{2}\right) + \frac{\pi}{4}\mathrm{Cl}_2\left(\frac{\sqrt{A}-1}{\sqrt{A+1}},\frac{\pi}{2}\right) - J(A+2). \tag{11.13}$$

This may again be further refined in terms of classical Clausen values.

11.3 The promised closed forms for J

$J(1)$. With this notation in hand, on setting $\theta := \arctan(1/\sqrt{2})$ we may rewrite $J(1)$ as follows

$$\begin{aligned} J(1) \quad &= \quad 2\theta\pi\,\log\frac{4}{3} + \frac{\pi^2}{16}\log 3 - \frac{7\pi^2}{8}\,\log 2 - \frac{1}{2}\,\log^2 2\log 3 + \log^3 2 - \frac{3}{2}\mathrm{G}\pi + \log^2(1+\sqrt{2})\log\frac{2}{3} \\ &+ \quad 2\,\mathrm{Li}_3\left(\sqrt{2},\frac{\pi}{4}\right) + \log\frac{8}{3}\,\mathrm{Li}_2\left(\frac{\sqrt{6}}{4},\theta\right) - 2\log\frac{4}{3}\,\mathrm{Li}_2\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}-\theta\right) + 2\theta\mathrm{Cl}_2\left(\frac{\sqrt{6}}{4},\theta\right) + 4\theta\mathrm{Cl}_2\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}-\theta\right) \\ &+ \quad \frac{\pi}{2}\mathrm{Cl}_2\left(\sqrt{6}+\sqrt{3},\theta\right) + \frac{\pi}{2}\mathrm{Cl}_2\left(\sqrt{6}-\sqrt{3},\theta\right) + \left(\frac{\pi}{4}+\theta\right)\mathrm{Cl}_2\left(\sqrt{3}+\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}+\theta\right) + \left(\frac{3\pi}{4}+\theta\right)\mathrm{Cl}_2\left(\sqrt{3}-\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}-\theta\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad 4\,\mathrm{Li}_3\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}-\theta\right) - 2\,\mathrm{Li}_3\left(\frac{\sqrt{6}}{4},\frac{\pi}{2}-\theta\right) + \mathrm{Li}_3\left(\sqrt{3}+\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}-\theta\right) + \mathrm{Li}_3\left(\sqrt{3}-\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}-\theta\right)\\
&+ \quad \left(\frac{1}{2}\log\frac{2}{3} - \log(1+\sqrt{2})\right)\mathrm{Li}_2\left(\sqrt{3}+\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}+\theta\right) + \left(\frac{1}{2}\log\frac{2}{3} + \log(1+\sqrt{2})\,\mathrm{Li}_2\left(\sqrt{3}-\frac{\sqrt{3}}{\sqrt{2}},\frac{\pi}{4}-\theta\right)\right).
\end{aligned}
$$

Moreover, (11.3) allows for all Cl_2 values to be expressed in terms of the classical Clausen function. We have placed the terms at the top for which we have thus shown reduction to elementary constants and classical polylogarithms or Clausen functions. Further simplification is possible of terms such as $\log\frac{8}{3}\,\mathrm{Li}_2\left(\frac{\sqrt{6}}{4},\theta\right)$ and $2\log\frac{4}{3}\,\mathrm{Li}_2\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}-\theta\right)$.

We now opt to set $\lambda := \log(1+\sqrt{2})$ and $\theta := \arctan\sqrt{2}$. We eventually arrive at:

$$
\begin{aligned}
J(1) \;=\;& \frac{\lambda^2}{2}\log\frac{3}{2} + \frac{\pi}{2}\left(\frac{\pi}{2}-\theta\right)\lambda + \pi\theta\log\frac{4}{3} - \frac{35}{32}\,\zeta(3) - \frac{1}{2}\log^3 2 + \frac{1}{4}\log^2 2\log 3 + \frac{9\pi^2}{32}\log\frac{3}{2}\\
-\;& \frac{\pi^2}{4}\log 2 + 2\,\mathrm{Li}_3\left(\frac{\sqrt{6}}{4},\theta-\frac{\pi}{4}\right) - 4\,\mathrm{Li}_3\left(\frac{\sqrt{3}}{2},\theta\right) + \mathrm{Li}_3\left(\sqrt{3}-\sqrt{\frac{3}{2}},\theta-\frac{\pi}{4}\right) + \mathrm{Li}_3\left(\sqrt{3}+\sqrt{\frac{3}{2}},\theta-\frac{3\pi}{4}\right)\\
+\;& \frac{\theta}{2}\,\mathrm{Cl}_2(4\theta-\pi) + (2\,\theta-\pi)\,\mathrm{Cl}_2(4\theta) - 3(2\theta-\pi)\,\mathrm{Cl}_2(2\theta) - \frac{\pi}{2}\mathrm{Cl}_2\left(2\theta-\frac{\pi}{2}\right) + \frac{\pi}{2}\,\mathrm{Cl}_2\left(2\theta+\frac{\pi}{2}\right) - G\,\pi.
\end{aligned}
$$

Each level of simplification reveals more structure. We now note that each Li_3 value is of the form $\mathrm{Li}_3(\frac{\sec\tau}{2},\tau)$ where $\tau := \theta + \frac{k\pi}{4}, 0 \le k \le 3$. Each of these may be resolved by an application of (11.8). This leads to:

$$
\begin{aligned}
J(1) \;=\;& \frac{\pi}{2}\,\mathrm{Cl}_2\left(2\theta-\frac{\pi}{2}\right) + \frac{\pi}{2}\,\mathrm{Cl}_2\left(2\theta+\frac{\pi}{2}\right) - \frac{\theta}{2}\,\mathrm{Cl}_2(4\theta-\pi) + (2\,\theta-\pi)\,\mathrm{Cl}_2(4\theta) - 3(2\theta-\pi)\,\mathrm{Cl}_2(2\theta)\\
+\;& 2\,\mathrm{Cl}_3(2\theta-\pi) - \mathrm{Cl}_3(2\theta) - \frac{1}{2}\,\mathrm{Cl}_3\left(2\theta+\frac{\pi}{2}\right) - \frac{1}{2}\,\mathrm{Cl}_3\left(2\theta-\frac{\pi}{2}\right) - G\,\pi - \frac{35}{32}\,\zeta(3). \qquad (11.14)
\end{aligned}
$$

J(2)**.** Surprisingly $J(2)$ is significantly simpler—because of the specific angles engaged. Though one has to carefully approach the removable singularity. This again leads to a very large symbolic expression which reduces to:

$$
\begin{aligned}
J(2) \;=\;& \frac{1}{108}\,\Psi'\left(\frac{1}{3}\right)\sqrt{3}\pi + \frac{1}{108}\,\Psi'\left(\frac{1}{6}\right)\pi\sqrt{3} - \frac{2}{81}\,\pi^3\sqrt{3} + \left(\frac{5}{12}\log(1+\sqrt{3}) + \frac{1}{24}\log 2\right)\pi^2\\
-\;& \frac{53}{48}\,\zeta(3) - \mathrm{G}\pi + \mathrm{Re}\,\mathrm{Li}_3\left(\frac{1-i+\sqrt{3}+i\sqrt{3}}{2(\sqrt{3}+1)}\right) + \mathrm{Re}\,\mathrm{Li}_3\left(\frac{1+i-\sqrt{3}+i\sqrt{3}}{2(\sqrt{3}-1)}\right) - \frac{\pi}{2}\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{i+\sqrt{3}}{\sqrt{3}-1}\right)\\
+\;& \frac{5\pi}{12}\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{-1+i+\sqrt{3}+i\sqrt{3}}{2(\sqrt{3}-1)}\right) - \frac{\pi}{12}\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{1+i+\sqrt{3}-i\sqrt{3}}{2(\sqrt{3}+1)}\right) + \frac{\pi}{2}\,\mathrm{Im}\,\mathrm{Li}_2\left(\frac{i+\sqrt{3}}{\sqrt{3}+1}\right).
\end{aligned}
$$

Again, the dilog terms can be nicely resolved as Clausen values and the trilogarithms can be manipulated as above. This produces:

$$
\begin{aligned}
J(2) \;=\;& \frac{\pi}{4}\sqrt{3}L_{-3}(2) + \frac{5}{12}\pi^2\log(1+\sqrt{3}) + \frac{1}{24}\pi^2\log 2 - \frac{53}{48}\zeta(3) - G\,\pi \qquad (11.15)\\
+\;& \mathrm{Li}_3\left(\frac{\sqrt{3}-1}{\sqrt{2}},\frac{\pi}{12}\right) - \mathrm{Li}_3\left(\frac{\sqrt{3}+1}{\sqrt{2}},\frac{5\pi}{12}\right) + \frac{\pi}{2}\,\mathrm{Cl}_2\left(\frac{\sqrt{3}-1}{2},\frac{\pi}{6}\right) - \frac{\pi}{2}\,\mathrm{Cl}_2\left(\frac{\sqrt{3}+1}{2},\frac{\pi}{6}\right)\\
+\;& \frac{5\pi}{12}\,\mathrm{Cl}_2\left(\frac{\sqrt{3}+1}{\sqrt{2}},\frac{5\pi}{12}\right) - \frac{\pi}{12}\,\mathrm{Cl}_2\left(\frac{\sqrt{3}-1}{\sqrt{2}},\frac{\pi}{12}\right).
\end{aligned}
$$

Remarkably (11.15) reduces to a single line:

$$J(2) \quad = \quad \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\,\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\,\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right). \tag{11.16}$$

We can use (11.12) to replace $\mathrm{Cl}_2\left(\frac{5\pi}{6}\right)$ by $\frac{4}{3}G - \mathrm{Cl}_2\left(\frac{\pi}{6}\right)$. We can also substitute $\mathrm{Cl}_2\left(\frac{\pi}{6}\right) = \frac{G}{3} + \frac{3\sqrt{3}}{16}L_{-3}(2)$, where $L_{-3}(2)$ is the primitive L-series modulo 3. Thus, alternatively we have

$$J(2) \quad = \quad \frac{5\pi}{16}\sqrt{3}\,L_{-3}(2) - \frac{\pi}{2}\,G - \frac{7}{48}\,\zeta\,(3) + \frac{\pi^2}{8}\,\ln 2. \tag{11.17}$$

.

$J(3)$**.** Using the decomposition in Theorem 5.1, an explicit polylogarithmic form of $\mathcal{K}_5 = J(3)$ was obtained in *Mathematica*. Said expression was roughly 30000 characters long—or 62000 when converted to a *Maple* expression. Either of these language forms comes down to about 50 kilobits of "expression entropy," as discussed in Section 6. The challenge is to find a much smaller expression. Write

$$J(3) = \mathcal{K}_5 = R_5 + C_5 + L_5 \tag{11.18}$$

where R_5 are the real trilog and dilog terms terms, C_5 are the pure Clausen terms, and L_5 collects the rest. Then, after a very large amount of symbolic work exploiting many formulas in Lewin [20], on setting $\lambda := \log\left(1+\sqrt{2}\right)$, $\theta := \arctan 2$ and $\tau := \arctan\sqrt{2}$, we obtain for R_5:

$$\begin{aligned}
R_5 \quad = \quad & \lambda\left(\mathrm{Li}_2\left(\frac{\sqrt{15}}{6} - \frac{\sqrt{30}}{12}, \theta+\tau\right) - \mathrm{Li}_2\left(\frac{\sqrt{15}}{6} + \frac{\sqrt{30}}{12}, \tau-\theta\right) - 2\,\mathrm{Li}_2\left(\frac{\sqrt{30}}{6}, \theta+\tau-\pi\right)\right) \\
+ \quad & 2\,\lambda\left(\mathrm{Li}_2\left(\frac{\sqrt{10}}{3}, \theta-\tau\right) - \mathrm{Li}_2\left(\frac{2}{\sqrt{3}}, \tau+\frac{3\pi}{4}\right) + \mathrm{Li}_2\left(\frac{2}{\sqrt{3}}, \tau-\frac{\pi}{4}\right) + \mathrm{Li}_2\left(\frac{\sqrt{30}}{6}, \tau-\theta\right) - \mathrm{Li}_2\left(-\frac{\sqrt{2}-1}{2}\right)\right) \\
+ \quad & 2\lambda\left(\mathrm{Li}_2\left(\frac{\sqrt{10}}{6} + \frac{\sqrt{5}}{6}, \pi-\theta-\tau\right) - \mathrm{Li}_2\left(\frac{\sqrt{10}}{3}, \theta+2\,\tau-\pi\right) - \mathrm{Li}_2\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}, \tau-\frac{\pi}{4}\right)\right) \\
+ \quad & \frac{1}{2}\left(\mathrm{Li}_2\left(\frac{\sqrt{15}}{6} - \frac{\sqrt{30}}{12}, \theta+\tau\right) + \mathrm{Li}_2\left(\frac{\sqrt{15}}{6} + \frac{\sqrt{30}}{12}, \tau-\theta\right)\right)\log\left(\frac{24}{5}\right) - \mathrm{Li}_3\left(\frac{\sqrt{10}}{6} - \frac{\sqrt{5}}{6}, \theta-2\,\tau\right) \\
- \quad & \mathrm{Li}_3\left(\frac{\sqrt{10}}{6} + \frac{\sqrt{5}}{6}, \theta+2\,\tau-\pi\right) - \mathrm{Li}_3\left(\frac{\sqrt{15}}{6} - \frac{\sqrt{30}}{12}, \theta+\tau\right) + \mathrm{Li}_3\left(\frac{\sqrt{15}}{6} + \frac{\sqrt{30}}{12}, \tau-\theta\right) \\
+ \quad & \mathrm{Li}_3\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}, \tau-\frac{3\pi}{4}\right) + \mathrm{Li}_3\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}, \frac{\pi}{4}-\tau\right) + \mathrm{Li}_3\left(\frac{\sqrt{10}}{6}, \frac{\pi}{4}-\theta\right) + \mathrm{Li}_3\left(\frac{\sqrt{10}}{2}, \frac{3\pi}{4}-\theta\right) \\
- \quad & 2\,\mathrm{Li}_3\left(\frac{\sqrt{5}}{2}, \theta\right) + \mathrm{Li}_3\left(-\frac{\sqrt{2}-1}{2}\right) - \mathrm{Li}_3\left(2\sqrt{2}-2\right).
\end{aligned}$$

Likewise, the constant and pure logarithmic coefficients L_5 reduce to:

$$\begin{aligned}
L_5 \quad = \quad & -\frac{7}{16}\,\zeta\,(3) - \frac{1}{2}\,G\,\pi + \frac{7}{6}\,\lambda^3 + \left(\frac{1}{2}\,\log 5 - \frac{1}{2}\log 3 - 2\log 2\right)\lambda^2 + \frac{3\pi}{2}\,\left(\log 2 - \frac{2}{3}\log 5 + \log 3\right)\tau, \\
+ \quad & \left(\theta^2 - 3\,\tau^2 + 3\,(4\,\tau-\pi)\,\theta - \frac{23}{48}\pi^2 + \frac{1}{4}\,\log^2 2 + \frac{3}{4}\log^2 3 - \log 3\log 5 + \frac{1}{4}\log^2 5 + \left(\frac{3}{2}\log 3 - \log 5\right)\log 2\right)\lambda \\
+ \quad & \left(\frac{1}{2}\log 3 - \frac{1}{2}\log 5 + \frac{3}{2}\log 2\right)\theta^2 - \frac{\pi}{2}\left(\frac{3}{2}\log 3 + \log 2\right)\,\theta - \left(\frac{7}{2}\log 2 - 2\log 5 + \frac{7}{2}\log 3\right)\tau^2 \\
+ \quad & 3\log^3 2 + \frac{5}{24}\log^3 3 - \frac{1}{8}\log^3 5 - \left(\frac{1}{2}\log 3\log 5 + 3\log^2 2 + \frac{3}{8}\log^2 5\right)\log 3
\end{aligned}$$

$$+ \quad \left(\frac{11}{8}\log^2 3 - \frac{9}{4}\log 3 \log 5 + \frac{9}{8}\log^2 5 - \frac{25}{8}\log 2 \log 5\right)\log 2 + \left(\frac{23}{6}\log 5 - \log 2\right)\frac{\pi^2}{16}$$

Also

$$\begin{aligned}
C_5 &= \theta\left(\frac{1}{2}\,\mathrm{Cl}_2\left(\pi - 4\,\theta\right) + 2\,\mathrm{Cl}_2\left(2\,\theta - \pi\right) - \mathrm{Cl}_2\left(4\,\tau + 2\,\theta\right) + \mathrm{Cl}_2\left(4\,\tau - 2\,\theta\right) - \mathrm{Cl}_2\left(2\,\tau - 2\,\theta\right) + \mathrm{Cl}_2\left(2\,\tau + 2\,\theta\right)\right) \\
&+ \frac{\pi}{4}\Bigg(2\mathrm{Cl}_2\left(2\,\theta - \pi\right) - 2\mathrm{Cl}_2\left(\frac{\pi}{2} + 2\,\theta\right) + \mathrm{Cl}_2\left(\frac{\pi}{2} - 2\,\theta\right) + 9\mathrm{Cl}_2\left(2\,\tau - \frac{\pi}{2}\right) - 9\mathrm{Cl}_2\left(2\tau + \frac{\pi}{2}\right) \\
&+ \mathrm{Cl}_2\left(4\,\tau - 2\,\theta\right) - \mathrm{Cl}_2\left(4\,\tau + 2\,\theta\right)\Bigg)..
\end{aligned}$$

Formula (11.18) was numerically checked to 200 decimal places. Given the final form for $J(1), J(2)$ it is suspected that $J(3)$ can be further simplified. Indeed, most of the Li_3 terms are susceptible to (11.8)—with consequent simplification of L_5.

$J(4)$. For $J(4)$ the corresponding expression has been reduced from roughly 100 kilobits of expression entropy, down to expressions similar to but somewhat longer than those for $J(3)$. We now use angles $\arctan\sqrt{5}$ and $\arctan\sqrt{15}$ and noninteger logarithms $\log\left(1+\sqrt{3}\right)$ and $\log\left(1+\sqrt{15}\right)$.

In each case the angles are coupled to those engaged in (11.11) for $A-1$.

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