MORE DESIGNER DECIMALS: THE INTEGERS AND THEIR GEOMETRIC EXTENSIONS

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Our adventure into the world of "designer decimals" began when the second author came across a curiosity on page 16 of Roberts' *Elementary Number Theory* [7], namely, $10000/9899 = 1.0102030508132134559\ldots$. He noticed that the difference between the numerator and denominator was 101. Inspired by this, he decided to try computing the decimal expansions of the fractions 100/89 and 1000000/998999 (the differences between numerator and denominator are 11 and 1001) and discovered a surprising result: both expansions begin with terms of the Fibonacci sequence! Tom Osler and the second author [9] went on to prove using generating functions that this class of fractions always produces decimal expansions containing terms of the Fibonacci sequence. Soon after, Bicknell-Johnson [1] found a more general class of fractions that give decimal expansions containing the initial terms of a sequence $\{a_n\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$a_n = pa_{n-1} + qa_{n-2}, \tag{1}$$

where p and q are integers. Fractions that generate the Fibonacci and Fibonacci-like sequences (sequences that satisfy recurrence (1) with $p = q = 1$, such as the Lucas numbers) have been studied by other authors as well; see, for example, [4], [5], and [6]. Bicknell-Johnson [2] also studied fractions where the Fibonacci sequence appears from right to left in the decimal expansion.

In the course of his investigations, the second author also found another curious fraction: 10000/9801. Note that the difference between the numerator and denominator here is 199. Similarly, the fractions 100/81 and 1000000/998001, corresponding to differences of 19 and 1999, are "designer decimals" as well. In this paper, we provide an arithmetical approach to find the properties of the fraction 10000/9801 and its relatives, without the use of generating functions. We also give other designer decimals that produce integer sequences which do not satisfy recurrence (1), and prove an interesting result on the remainders of these designer decimals when one performs division on them.

The Fraction $\dfrac{10000}{9801}$ and its Generalizations

The decimal expansion of the fraction $\frac{10000}{9801}$ begins $1.0203040506\ldots$. A few questions arise naturally to the inquisitive reader: "Do all the integers from 1 to 99 occur in sequence?" "Since we have a fraction, the decimal expansion must repeat. What is the length and nature of the repeating part?" In this section, we answer both of these questions.

The key to unraveling the fraction $\frac{10000}{9801}$ is to notice that $9801 = 99^2$. Therefore, we may express it as

$$\frac{10000}{9801} = \left(\frac{100}{99}\right)^2 = (1.0101010101\ldots)^2.$$

Now we use a trick that may be familiar to those who know the classical algorithm for computing the square root of a number. That is, we take a decimal $(a_0.a_1a_2\ldots)^2$ and expand it as $(a_0 + 10^{-1}(a_1.a_2\ldots))^2 = (a_0^2 + 0.2 \times a_0 \times a_1) + 10^{-2}(a_1.a_2a_3\ldots)^2$. Therefore, we expand the fraction as follows.

$$\begin{aligned}
\left(\frac{100}{99}\right)^2 &= (1 + \frac{1}{100}(1.\overline{01}))^2 = 1 + 2(0.\overline{01}) + \frac{1}{10000}(1.\overline{01})^2 \\
&= 1.02020202\ldots + \frac{1}{10000}(1.020202\ldots + \frac{1}{10000}(1.0202\ldots + \cdots)) \\
&= \quad 1.0202\ 0202\ 0202\ 0202\ldots \\
&\quad + 0.0001\ 0202\ 0202\ 0202\ldots \\
&\quad + 0.0000\ 0001\ 0202\ 0202\ldots \\
&\quad + 0.0000\ 0000\ 0001\ 0202\ldots \\
&\quad + \ldots \\
&= \quad 1.0203\ 0405\ 0607\ 0809\ldots.
\end{aligned}$$

After 49 steps we find the decimal expansion of $\frac{10000}{9801}$ to be $1.0203\ldots 969\overline{798}+ (0.0001)^{-49}(1.\overline{01})^2$. The next step gives us

$$\begin{aligned}
\left(\frac{100}{99}\right)^2 &= \quad 1.02\ldots 9697\ 9898\ 9898\ 9898\ldots \\
&\quad + 0.00\ldots 0000\ 0001\ 0202\ 0202\ldots \\
&\quad + \ \ldots \\
&= \quad 1.02\ldots 9697\ 9900\ 0101\ 0101\ldots \\
&\quad + 0.00\ldots 0000\ 0000\ 0001\ 0202\ldots \\
&\quad + \ \ldots \\
&= \quad 1.02\ldots 9697\ 9900\ 0102\ 0304\ldots.
\end{aligned}$$

Continuing this for another 49 steps, we find that

$$\begin{aligned}\left(\frac{100}{99}\right)^2 &= \quad 1.02\ldots 9697\ 9900\ 0102\ \ldots\ 9798\ 9999\ 9999\ldots \\ &\quad + 0.00\ldots 0000\ 0000\ 0000\ \ldots\ 0000\ 0001\ 0202\ldots \\ &\quad + \ldots \\ &= \quad 1.02\ldots 9697\ 9900\ 0102\ \ldots\ 9799\ 0001\ 0202\ldots \\ &\quad + \ldots\end{aligned}$$

That is,

$$\left(1.\overline{01}\right)^2 = 1.02\ldots 969799000102\ldots 9799 + \left(\frac{1}{10000}\right)^{-98}(1.\overline{01})^2.$$

Therefore, the repeating part is $0203\ldots 97990001$. In other words,

$$\left(\frac{100}{99}\right)^2 = 1.\overline{02\ldots 9697990001}.$$

We note that this analysis can be made for all fractions of the type

$$\begin{aligned}Q_n :&= \left(\frac{10^n}{10^n-1}\right)^2 = (1.\overline{\underbrace{0\ldots0}_{n-1}1})^2 = q_n^2 \\ &= 1 + 2(10^{-n}q_n) + 10^{-2n}q_n^2. \qquad (2)\end{aligned}$$

We summarize our findings in Theorem 1. For a non-negative integer i, let the *n-padded representation of i* be the integer i written with sufficient leading zeros so that it has length n. For example, the 3-padded representations of 0, 3, and 15 are 000, 003, and 015 respectively.

Theorem 1. *Given a positive integer n, let a_i denote the n-padded representation of the non-negative integer i. Then the decimal expansion of Q_n is*

$$1.\overline{a_2a_3\ldots.a_{10^n-3}a_{10^n-1}a_0a_1}.$$

We also have the following curious consequence.

Corollary. *For a fixed positive integer n, the smallest positive t such that*

$$10^t - 1 \equiv 0 \,(\mathrm{mod}\,(10^n-1)^2)$$

is $t = n(10^n - 1)$. That is, the shortest string of 9s that is divisible by the square of a string of n 9s has length $n(10^n - 1)$.

Proof. Note that $\frac{P}{10^n-1} = 0.\overline{P'}$, where P' is the n-padded representation of P. If $10^t - 1 \equiv 0 \,(\mathrm{mod}\, b)$, then $10^t - 1 = bm$ for some m. So

$$\frac{1}{b} = \frac{m}{10^t - 1}$$

and thus t must be a multiple of the minimum length of the repeating part of the decimal expansion of $1/b$. Therefore, the minimum t for which $10^t - 1 \equiv 0 \pmod{(10^n-1)^2}$ is the minimum length of the repeating part of $\left(\frac{10^n}{10^n-1}\right)^2$. By Theorem 1, the minimum repeating part is exactly the sequence $\{a_2, a_3, ..., a_{10^n-3}, a_{10^n-1}, a_0, a_1\}$. Since each term has length n, and there are $10^n - 1$ terms, we conclude that $t = n(10^n - 1)$. $\square$

To find out more about periodicity, period length, and other related topics, see Yates' book [10].

Fractions of the Form $\left(\frac{10^n}{10^n-1}\right)^k$

Another way to show that the fractions $\left(\frac{10^n}{10^n-1}\right)^2$ yield the sequence of integers in their decimal expansions is to consider what happens when we multiply a number by

$$\frac{10^n}{10^n-1} = 1.\underbrace{\overline{0...01}}_{\text{length } n}.$$

Let $x = 1.a_2a_3a_4....$ be the decimal expansion of some number, divided into blocks of length n. That is, each a_i is the n-padded representation of an integer (which we also call a_i, for simplicity) such that $0 \leq a_i < 10^n$. In other words, if we set $a_1 = 1$, then

$$x = \sum_{i=0}^{\infty} a_{i+1} 10^{-in}.$$

Consider the product

$$\begin{aligned} x\frac{10^n}{10^n-1} &= \left(\sum_{i=0}^{\infty} a_{i+1}10^{-in}\right)\left(\sum_{j=0}^{\infty} 10^{-jn}\right) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i+1}10^{-(i+j)n}. \end{aligned}$$

Let $k = i + j$, so that

$$\begin{aligned} x\frac{10^n}{10^n-1} &= \sum_{k=0}^{\infty} 10^{-kn} \sum_{i=0}^{k} a_{i+1} \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=1}^{k+1} a_i\right)10^{-kn} \\ &=: \sum_{k=0}^{\infty} b_k 10^{-kn}. \end{aligned}$$

Therefore, multiplying x by $\frac{10^n}{10^n-1}$ produces a decimal expansion where the kth block of n digits is equal to b_k (which is 1 plus the sum of the first k blocks of n digits in the decimal expansion of x) unless b_k has more than n digits. Suppose this overflow occurs at the Kth block. Then the decimal formed by the remaining blocks have size

$$\begin{aligned}E &:= \sum_{k=K+1}^{\infty}\left(\sum_{i=1}^{K} a_i + \sum_{i=K+1}^{k} a_i\right)10^{-kn} < \sum_{k=K+1}^{\infty}(10^n + (k-K))10^n)10^{-kn}\\ &= \sum_{k=K+1}^{\infty}(k-(K+1)+2)10^{-(k-(K+1))n-Kn} = 10^{-Kn}\sum_{\ell=0}^{\infty}(\ell+2)10^{-\ell n},\end{aligned}$$

because $a_i < 10^n$ for all i and $\sum_{i=1}^{K} a_i = b_{K-1} < 10^n$. Since $\ell + 2 < 10^\ell$ for $\ell \geq 1$, we find that

$$\begin{aligned}E &< 10^{-Kn}\left(2 + \sum_{\ell=1}^{\infty} 10^{-\ell(n-1)}\right)\\ &= 10^{-Kn}\left(2 + \frac{10^{-(n-1)}}{1-10^{-(n-1)}}\right) < 3 \times 10^{-Kn}.\end{aligned}$$

Therefore, any further carryover from the remaining blocks adds at most 3 to b_K. Since $b_K = b_{K-1} + a_{K+1}$ and both summands are less than 10^n, we have $b_K < 2 \times 10^n$. Therefore, there is at most a carryover of 2 to the $(K-1)$st block, b_{K-1}, and thus $b_{K-1} + 2 < 10^n + 1$. This means there may be a carry-over of 1 to the $(K-2)$nd block, b_{K-2}.

Note that the only way there can be further carryover affecting earlier blocks (b_{K-3}, b_{K-4}, etc.) is if $b_{K-2} = 10^n - 1$. But since $10^n > b_{K-1} \geq b_{K-2} \geq b_{K-3} \cdots$, there is further carryover only if $b_{K-1} = b_{K-2}$. In other words, the carryover affects blocks beyond b_{K-2} only if $b_{K-1} = 10^n - 1$ and $a_K = 0$.

Thus, the fraction $(\frac{10^n}{10^n-1})^2$ begins with the sequence of positive integers, starting at 2, with each integer occupying a block of length n, and up to at least $10^n - 3$ (that is, $K = 99$) before the pattern will break down. If we multiply our fraction by $\frac{10^n}{10^n-1}$, we obtain blocks of length n consisting of sums of the positive integers. As we have just seen, each time we multiply by $\frac{10^n}{10^n-1}$, we get sums of the blocks of length n from the previous fraction. Clearly, with each additional factor, the sequence breaks down earlier and earlier. However, for a fixed number of factors, we can choose a large enough n so that the sequence breaks down as late as we wish.

Now, all that's left to do is to determine what exactly is the sequence we get from this procedure. It turns out that these are the higher-dimensional pyramidal numbers. The nth k-dimensional pyramidal number $p_k(n)$ is defined to be the sum of the first n $(k-1)$-dimensional pyramidal numbers, with the 0-dimensional pyramidal numbers defined as the sequence consisting entirely of 1s. That is, for $n \geq 0$,

$$p_k(n) := \begin{cases} \displaystyle\sum_{j=1}^{n} p_{k-1}(j), & \text{if} \quad k > 0,\\ 1, & \text{if} \quad k = 0.\end{cases}$$

Note that the 1-dimensional pyramidal numbers are the positive integers and the 2-dimensional ones are the triangular numbers. The 3-dimensional pyramids with edge length n are formed by stacking together triangles of edge length less than or equal to n. Specific values for these sequences can be found in Sloane and Plouffe's *Encyclopedia of Integer Sequences* [8]. One can easily verify, by double induction, that

$$p_k(n) = \binom{n+k-1}{k}. \tag{3}$$

Now that we know what the sequences in the decimal expansions of these fractions are, we can formulate our theorem.

Theorem 2. *For fixed positive integers k and ℓ, there exists an integer n for which the decimal expansion of*

$$\left(\frac{10^n}{10^n-1}\right)^k$$

begins with

$$1.a_2a_3\ldots a_\ell,$$

where a_ℓ is the n-padded representation of $p_{k-1}(\ell)$, provided that it has fewer than n digits.

Powers of n and the Fibonacci Sequence

As we stated earlier, Osler and the second author [9] used generating functions to show that the fractions $\frac{10^{2n}}{10^{2n}-10^n-1}$ produced the Fibonacci sequence in their decimal expansions, and Bicknell-Johnson [1] generalized these fractions to give sequences of the form (1). Using this tool, we may just as easily find fractions that produce powers of n in their decimal expansions.

Theorem 3. *For any positive integers b and ℓ, there exists an integer n for which the decimal expansion of $\frac{10^n}{10^n-b}$ begins with the n-padded representations of the first ℓ powers of b.*

Proof. The generating function for the sequence of powers of b is

$$g_b(x) = \sum_{k=0}^{\infty} b^k x^k = \sum_{k=0}^{\infty} (bx)^k = \frac{1}{1-bx},$$

and this converges for $|x| < b$.

For sufficiently large n, $x = 10^{-n}$ yields an expansion in blocks of size n. The last equality shows that

$$g_b(10^{-n}) = \frac{1}{1-b10^{-n}} = \frac{10^n}{10^n-b}.$$

□

We conclude with two curious observations about the remainders of these fractions when we perform division.

Theorem 4. *If b is not a multiple of 10, then when 10^n is divided by $10^n - b$ in blocks of size n, the remainder at step ℓ, up to the $\lfloor \frac{n+1}{\log_{10} b} \rfloor$th step, is b^ℓ.*

Proof. We know that

$$\frac{10^n}{10^n - b} = 1.00\ldots b00\ldots b^2 00\ldots b^3 \ldots = 1 + \frac{b}{10^n - b}.$$

Now notice that

$$\frac{b^\ell}{10^n - b} - \frac{b^\ell}{10^n} = \frac{b^\ell(10^n) - b^k(10^n - b)}{(10^n)(10^n - b)} = \frac{b^{\ell+1}}{(10^n)(10^n - b)}.$$

Therefore,

$$\begin{aligned}\frac{10^n}{10^n - b} &= 1 + \frac{b}{10^n - b}\\ &= 1 + \frac{b}{10^n} + \frac{b^2}{10^n(10^n - b)}\\ &= 1 + \frac{b}{10^n} + \frac{b^2}{10^{2n}} + \frac{b^3}{10^{2n}(10^n - b)}\\ &= \cdots .\end{aligned}$$

Here the numerator of the last term is the remainder. This division works as long as b^ℓ has no more than n digits. That is, until $b^\ell = 10^{n+1}$, or $\ell = \lfloor \frac{n+1}{\log_{10} b} \rfloor$. □

There is a similar result for the Fibonacci fractions.

Theorem 5. *When 10^{2n} is divided by $10^{2n} - 10^n - 1$ in blocks of size n, the remainder at step ℓ is $10^n F_{\ell+1} + F_\ell$, where F_ℓ is the ℓth Fibonacci number (with $F_0 = 0$, $F_1 = F_2 = 1$). This pattern holds as long as $10^n F_{\ell+1} + F_\ell < 10^{2n}$.*

Proof. Consider the difference

$$x_i = \frac{10^n F_{i+1} + F_i}{10^{2n} - 10^n - 1} - \frac{F_{i+1}}{10^n}.$$

Simplifying, we obtain

$$\begin{aligned}x_i &= \frac{10^n F_i + 10^n F_{i+1} + F_{i+1}}{(10^n)(10^{2n} - 10^n - 1)}\\ &= \frac{10^n F_{i+2} + F_{i+1}}{(10^n)(10^{2n} - 10^n - 1)}\\ &= \frac{F_{i+2}}{10^{2n}} + \frac{x_{i+1}}{10^n}.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
\frac{10^{2n}}{10^{2n}-10^n-1} &= 10^n \frac{10^n F_1 + F_0}{10^{2n}-10^n-1} \\
&= 10^n \left(\frac{F_1}{10^n} + x_0\right) \\
&= 10^n \left(\frac{F_1}{10^n} + \frac{F_2}{10^{2n}} + \frac{x_1}{10^n}\right) \\
&= \cdots \\
&= 10^n \left(\frac{F_1}{10^n} + \cdots + \frac{F_\ell}{10^{\ell n}} + \left(\frac{F_{\ell+1}}{10^{(\ell+1)n}} + \frac{x_\ell}{10^{\ell n}}\right)\right) \\
&= F_1 + \frac{F_2}{10^n} + \cdots + \frac{F_\ell}{10^{(\ell-1)n}} + \frac{10^n F_{\ell+1} + F_\ell}{10^{(\ell-1)n}(10^{2n}-10^n-1)}.
\end{aligned}$$

Again, the numerator of the last fraction is the remainder at the ℓth step, and the pattern holds as long as that number has length less than $2n$. □

Example. The fraction

$$\begin{aligned}
\frac{10000}{9899} &= 1.01 + \frac{0201}{100(9899)} \\
&= 1.0102 + \frac{0302}{10^4(9899)} \\
&= 1.010203 + \frac{0503}{10^6(9899)} \\
&= 1.010203050813 + \frac{2113}{10^{12}(9899)} = \cdots.
\end{aligned}$$

Exercises

1. Let k be a non-zero digit, and let $a_0 = a_1 = 1$, $a_n = a_{n-1} + ka_{n-2}$ for $n \geq 2$. Show that if $f(x)$ is the generating function of $\{a_n\}_{n=0}^{\infty}$. (That is, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f(x) = \frac{1}{1-x-kx^2}$.)
2. Find the designer decimals that correspond to each of the sequences in Exercise 1, and show that the difference between the numerator and denominator is $10...0k$.
3. Are there any special sequences that correspond to other numerator-denominator differences? How do the numerator-denominator differences change when we use different values for a_0 and a_1?

4. Euler [3, p.44] computed the sum of the reciprocals of the first N k-dimensional pyramidal numbers. Use the identity
$$\frac{1}{n(n+k-1)} = \frac{1}{k-1}\left(\frac{1}{n} - \frac{1}{n+k-1}\right)$$
to show that the sum
$$\sum_{n=1}^{N} \frac{1}{p_k(n)}$$
is telescoping for each $k \geq 2$, and evaluate it in closed form.

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