MATH 3070
Assignment # 6 Solutions
Due Thursday, November 27, 2008

1. Let $x = [2, \overline{1, 1, 1, 4}]$ and $y = [\overline{1, 1, 1, 4}]$. Then we find that $(((y-1)^{-1}-1)^{-1}-1)^{-1}-4 = 1/y$. Simplifying, we obtain

$$\frac{1}{y} = \left(\frac{y-1}{2-y} - 1\right)^{-1} - 4 = \frac{2-y}{2y-3} - 4 = \frac{14-9y}{2y-3},$$

and so,

$$\begin{aligned} 2y - 3 &= 14y - 9y^2, \\ 9y^2 - 12y - 3 &= 0, \\ 3y^2 - 4y - 1 &= 0. \end{aligned}$$

Thus $y = \dfrac{4 + \sqrt{16+12}}{6} = \dfrac{2+\sqrt{7}}{3}$. (We took the positive root).

Since $x - 2 = 1/y$, we find that $x = 2 + \dfrac{3}{2+\sqrt{7}} = 2 + \dfrac{6 - 3\sqrt{7}}{4-7} = \sqrt{7}$.

2. Taking floors and flipping (using $\pi^2 \approx 9.8696044$), we find that $\pi^2 = [9, 1, 6, 1, 2, 47, 1, \ldots]$. Applying the recurrence, we obtain

$$\begin{aligned} p_0 = 9, p_1 = 10, p_2 = 6(10) + 9 = 69, p_3 = 79, p_4 = 2(79) + 69 = 227, p_5 = 10748, p_6 = 10975 \\ q_0 = 1, q_1 = 1, q_2 = 6(1) + 1 = 7, q_3 = 8, q_4 = 2(8) + 7 = 23, q_5 = 1089, q_6 = 1112. \end{aligned}$$

Thus the convergents are

$$9, 10, \frac{69}{7}, \frac{79}{8}, \frac{227}{23}, \frac{10748}{1089}, \frac{10975}{1112}.$$

3. If d is a perfect square, say $d = n^2$ with $n \geq 0$, then $x^2 - n^2y^2 = (x + ny)(x - ny) = 1$. If there is a solution (x, y) in integers, then $(\pm x, \pm y)$ is also a solution and thus we may assume both x and y are non-negative. This means that $x + ny \in \mathbb{N}$ (it cannot be zero) and thus $x + ny = 1$. If $n = 0$ then $(1, y)$ is a solution for any integer y. If $n > 0$ then we must have $x + ny = 1$ and $x - ny = 1$. Solving this pair of equations yields $x = 1$ and $y = 0$.

4. We easily find (by calculation or looking up in the text) that $\sqrt{5} = [2, \overline{4}]$. Thus the smallest (non-trivial) solution corresponds to the convergent at the end of a full period whose index is odd; that is, p_1/q_1. Therefore the smallest solution is $x = 2(4) + 1 = 9, y = 4$. The rest of the solutions are generated by taking $(9 + 4\sqrt{5})^n$ for $n = 2, 3, 4, 5$. This gives us

$$161 + 72\sqrt{5}, 2889 + 1292\sqrt{5}, 51841 + 23184\sqrt{5}, 930249 + 416020\sqrt{5}.$$

So the solutions are

$$(\pm 1, 0), (\pm 9, \pm 4), (\pm 161, \pm 72), (\pm 2889, \pm 1292), (\pm 51841, \pm 23184), (\pm 930249, \pm 416020).$$

5. Solution 1: We apply the pigeonhole principle. Fix N. Consider the numbers $y_n = nx - \lfloor nx \rfloor$ for $n = 1, \dots, N+1$. Since x is irrational, the y_n are irrational and lie in the open interval $(0,1)$. Subdivide this interval into N pieces of length $1/N$. That is, $(0,1) = (0,1/N] \cup (1/N, 2/N] \cup \cdots \cup ((N-1)/N, 1)$. By the pigeonhole principle some pair y_k, y_ℓ must lie in the same interval, so that $|y_k - y_\ell| = |kx - \ell x - \lfloor kx \rfloor + \lfloor \ell x \rfloor| = |qx - p| < 1/N$ for some integer p and $q = |k - \ell| < N$.

   Solution 2: Fix N, since the denominators q_n of the convergents to the continued fraction expansion of x are strictly increasing, we can find an n such that $q_n > N$. Thus $|x - p_n/q_n| < 1/q_n^2 < 1/q_n N$. Clearing denominators we find that $|q_n x - p_n| < 1/N$.