MATH 3070
Practice Problems

Note: This is a list of problems that hopefully will help reinforce some of the key ideas in the course. It is in no way an exhaustive accounting of examinable topics. Please refer to the notes as well as the homework for further study of topics not covered here.

1. Let a, b, x, y be integers such that $ax + by = n$. Prove that the product $\gcd(a, b) \gcd(x, y)$ divides n.

2. Prove that the product of any four consecutive integers is divisible by 24.

3. Prove that the product of any k consecutive integers is divisible by $k!$.

4. Prove that if Fermat's Last Theorem is true for and odd primes and 4, then it is true for general n. (FLT states that for all integers $n \geq 3$, the equation $x^n + y^n = z^n$ has no solutions in non-zero integers x, y, z.)

5. Find a triple of integers (x, y, z) such that $24x + 15y + 10z = 1$.

6. In base 10, the "sum-of-digits" divisibility rule works for 3 and 9. That is, n is divisible by 3 (resp. 9) if and only if the sum of its base 10 digits is divisible by 3 (resp. 9). When does this rule work in hexadecimal (base 16)?

7. Professor Evil has some poisoned candy. He split it up into three equal piles to bring to class on three separate days. On the first day, his candy divided up evenly among his 10 students. On the second day, seven students survived and so there were 2 pieces left over. So Prof. Evil added those 2 pieces to the pile for the third day, and it again divided evenly among the three remaining students. What is the minimum total amount of candy Prof. Evil had at the beginning?

8. Prove that there are infinitely many positive integers that cannot be expressed as the sum of six 4th-powers.

9. Fix a positive integer n. Define $f(x) = \gcd(n, x)$ for positive integers x. Prove that $f(x)$ is multiplicative but not necessarily completely multiplicative. For what n is f completely multiplicative?

10. Express $\varphi(n^k)$ in terms of $\varphi(n)$.

11. Let n be odd. Prove that if n is deficient, then so is $2^k n$.

12. Does $6x^2 \equiv 25 \pmod{37}$ have any solutions x?

13. Let p be a prime with $p \equiv 2 \pmod 3$. Prove that every integer is a cubic residue mod p.

14. Let p be an odd prime and let r and s be primitive roots mod p. Prove that rs is a quadratic residue mod p.

15. Prove that $\operatorname{ord}_n(x^k) = \dfrac{\operatorname{ord}_n(x)}{\gcd(k, \operatorname{ord}_n(x))}$.

16. Let x be rational. Prove that there is some positive integer N such that for every rational p/q in lowest terms with denominator $q > N$, one has $|x - p/q| > 1/q^2$.

17. Calculate the first 4 convergents of $\zeta(3) = 1.2020569031595942854\ldots$

18. Find 3 non-trivial solutions to the Pell equation $x^2 - 11y^2 = 1$.