# MATH 3070
## Assignment # 4
## Due Thursday, October 23, 2008

1. Find all values of $n$ such that $\varphi(n) = 12$.

2. Fix $k \geq 2$. It is easy to see that $n = 2^{k+1}$ satisfies $\varphi(n) = 2^k$. Find two other values of $n$ that satisfies that equation.

3. What is the remainder when $18! + 25!$ is divided by 23?

4. (a) Prove that numbers of the form $4n^2 + 1$ is never divisible by primes of the form $4k + 3$.
   (b) Use part (a) to show there are infinitely many primes of the form $4k + 1$.

5. For odd primes $p$, evaluate:
   (a) $\displaystyle\sum_{k=1}^{p-1} \left(\frac{k}{p}\right)$,
   (b) $\displaystyle\prod_{k=1}^{p-1} \left(\frac{k}{p}\right)$,

   where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

6. Prove that $a$ is a quadratic residue mod $p$ if and only if $a^{-1}$ is a quadratic residue mod $p$.

7. Let $p$ be a prime. When is 3 a quadratic residue mod $p$? When is 5 a quadratic residue mod $p$?

8. Interpret a rational $\frac{a}{b} \pmod{p}$ as $ab^{-1} \pmod{p}$ if $b \not\equiv 0 \pmod{p}$. Prove that

$$1 + \frac{1}{2^2} + \cdots \frac{1}{(p-1)^2} \equiv 0 \pmod{p}$$

   if $p > 3$ is prime.

   (Hint: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.)