MATH 2113 / CSCI 2113
Lecture Notes
Friday, 5 January 2007 to Monday 8 January, 2007

1. Enumerative Combinatorics

1.1. **Introduction.** We begin with an introduction to enumeration.

What is enumerative combinatorics? One could say that the subject is all about learning how to count (cleverly). On one level, counting is easy. If we want to count the number of objects that satisfy some given property, we can simply list them all, and then count. Of course, in most cases, that takes far too long. So, our goal is to find ways to help us count things *without* listing them all.

Most of you already do this. Consider the following example.

Example 1. *How many integers are in the interval* $[1000, 9999]$*?*

One solution would be to compute $9999 - 1000 + 1 = 9000$. If we consider each integer to be a point on the line, then the operation $9999 - 1000$ counts the number of gaps between consecutive integers from 1000 to 9999. We add one because the number of integers $= 1+$ number of gaps. (For example, $\{1, 2, 3\}$ contains 2 gaps and 3 integers.)

Let's do a slightly harder problem.

Example 2. *How many integers not containing the digit* 3 *are there in the interval* $[1000, 9999]$*?*

It is clear that this problem is somehow related to the first problem, but it is not immediately obvious how we can extend the above solution to solve this one. Perhaps we should try to find a different solution that works for both of these problems.

Let's go back to Example 1, and consider the count from a different perspective. An integer in the interval $[1000, 9999]$ has exactly 4 digits, and the leftmost digit cannot be 0. Also note that the interval covers every 4-digit number. So, we're actually counting the number of ways to place the digits 0–9 into 4 spots, with the leftmost (say, spot 1) chosen from the digits 1–9.

Now, there are 9 choices for the leftmost spot, and for each of those, there are 10 choices for the second spot. So there are $9 \times 10 = 90$ ways to choose digits for the 2 leftmost spots. For each of those 90 ways, there are 10 choices for the third spot, so there are $90 \times 10 = 900$ ways to choose digits for the three leftmost spots. Finally, there are 10 choices for the rightmost spot, and so there are $900 \times 10 = 9000$ integers in the interval.

It should be easy to see how to solve Example 2. We're counting 4-digit numbers, but we exclude the digit 3 from our choices. So we have $8 \times 9 \times 9 \times 9 = 5832$ integers satisfying the given property.

<make sure the class understands why we take a product in the above examples>

Note 1. *As in Example 1, there is often more than one way to solve a counting problem. Different ways of counting the same thing yields different insights into the structure of what's being counted.*

Note 2. *The problem in Example 2 can also be solved using a counting argument similar to the initial solution of Example 1. We might use an inclusion-exclusion type argument, take the 9000*

4-digit numbers we counted initially, and subtract off the number of 4-digit numbers that begin with a 3*, have a* 3 *in the second position, etc. (and make sure we don't count the same thing twice).*

Note 3. *Often, a problem that seems difficult simply needs to be rephrased, as in Example 2.*

1.2. **Arrangements and Selections.** Many counting problems can be reworded into problems involving either an arrangement or a selection problem. Therefore, it is worthwhile to write down an explicit formula for them.

Arrangements of Distinct Objects

Example 3. *In a race with 10 runners, gold, silver, and bronze medals are awarded to the first, second, and third place finishers, respectively. How many possible outcomes are there to the race? (that is, how many different ways are there to give out the medals?)*

In the above example, we want to arrange 10 distinct people in 3 positions. This can be solved in the same way as Example 2. There are 10 possible ways to pick the first place. For *each* of those choices, there are **9** possible runners for second place. So there are $10 \times 9 = 90$ ways to choose the top 2 runners. For *each* of these there are 8 choices left for 3rd place. So the number of outcomes is $10 \times 9 \times 8 = 720$.

In general, if we want to arrange n **distinct** objects in r positions, we have n choices for the first position, $n-1$ choices for the second position, etc. So we have the following theorem.

Theorem 1. *There are*

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} =: \ {}_nP_r$$

ways to arrange n distinct objects in r positions.

Note 4. *${}_nP_r$ is also called the number of permutations of n distinct objects in r positions.*

Selections of Distinct Objects

Example 4. *In lotto 6/49, six numbers are chosen, without replacement, from the set $\{1, 2, \ldots, 49\}$. How many possible tickets are there?*

This problem involves *selecting* 6 objects from a set of 49 distinct objects. Let's try to reduce this to the previous problem. If we draw the numbers one at a time, and place them in labelled bins (say from A to F), and count these, we get the number of arrangements of 49 objects into 6 positions. Of course, we're over-counting, because picking 1-2-3-4-5-6 gives the same outcome as picking 2-1-5-3-4-6. In fact, we've overcounted by exactly the number of internal arrangements of the six numbers on each ticket. That is, we've counted every ticket exactly ${}_6P_6 = 6!$ times. Therefore, we can divide out by 6! and we find that there are

$$\frac{{}_{49}P_6}{6!} = \frac{49!}{43!6!} = 13983816$$

possible tickets.

In general, if we want to select r objects from a set of n distinct objects, we can count using the above method. Placing the r objects into labelled bins reduces the problem to computing ${}_nP_r$. But we've counted every selection exactly $r!$ times, and so we divide out by $r!$. The formula is given in the following theorem.

Theorem 2. *The number of ways to select r objects from a set of n distinct objects is*

$$\frac{1}{r!}{}_nP_r = \frac{n!}{r!(n-r)!} =: {}_nC_r = \binom{n}{r}.$$

Note 5. ${}_nC_r$ *is also called the number of combinations of r objects from a set of n distinct objects.*

Note 6. $\binom{n}{r}$ *is read "n choose r" or "the binomial coefficient n choose r".*

Corollary 2.1. *(The Binomial Theorem)*

For every positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Since

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}},$$

each term in the resulting sum has either an x or y factor from each of the $x+y$ terms. Therefore, every term is of the form $x^k y^{n-k}$ for $0 \leq k \leq n$. For a fixed k, the number of such terms is the number of ways you can select k of the $x+y$ factors to contribute an x. Since you're counting the number of ways to select k factors from a total of n factors (that are treated as distinct from each other), there are $\binom{n}{k}$ ways to do it. □

Here's a number theoretic result that comes from our formula for selections.

Corollary 2.2. *The product of any n consecutive positive integers is divisible by $n!$.*

Proof. Let $(m+1)(m+2)\cdots(m+n)$ be a product of n consecutive positive integers (so $m \geq 0$). Then the quotient

$$\frac{(m+1)\cdots(m+n)}{n!} = \frac{(m+n)!}{m!n!} = \binom{m+n}{n}$$

is an integer, since it counts the number of ways to select n objects from a total of $m+n$ objects. □

Let us investigate a few more properties of the binomial coefficients.

Now, notice that the number of ways to pick k objects out of a bag of n objects, is the same as picking $n-k$ objects to leave in the bag. (That is, picking which $n-k$ object to leave in the bag automatically gives you a set of k objects to take out). Therefore, we have

Corollary 2.3.

$$\binom{n}{k} = \binom{n}{n-k}$$

This is also obvious from either the formula or the binomial theorem.

The argument we gave to prove Corollary 2.3 is an example of a *combinatorial proof* of an identity. We've proved it by counting the same thing in two different ways. A proof using the formula is an *analytic* or *algebraic proof.* Depending on the situation, one method would be preferred over the other. Later in the course we will see some identities that are easily proved combinatorially but difficult to show analytically.

Corollary 2.4. *(Pascal's Triangle)*
For $n \geq 2$, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. We can interpret the left-hand side is the number of ways to choose k numbers from the set $\{1, 2, \ldots, n\}$. Now, let us count this more precisely. For any choice of k numbers, either n is chosen or it is not. The total number of ways to choose k numbers will then be the number of ways to choose them, if n is chosen, plus the number of ways if n is not chosen. If n is chosen, then you're left with choosing $k-1$ numbers from the set $\{1, 2, \ldots, n-1\}$, and there are $\binom{n-1}{k-1}$ ways to do it. If n is not chosen, then all k numbers are from the set $\{1, 2, \ldots, n-1\}$ and there are $\binom{n-1}{k}$ ways to pick them. This implies the result. □

Notation. We will be using the set $\{1, 2, \ldots, n\}$ often, so it will be convenient to assign a symbol to it. We let $[n] = \{1, 2, \ldots, n\}$ for $n \geq 1$ and $[0] = \emptyset$.

Corollary 2.5. *Let S be a set with $|S| = n$. Then we have*

1. *The number of subsets of S with exactly k elements is $\binom{n}{k}$.*
2. *$|P(S)| = 2^n$, where $P(S)$ is the power set of S.*

Proof. Part 1 basically tells you how to count the subsets in order to derive part 2.

1. A subset of S with exactly k elements corresponds to a selection of k elements out of the n elements of S. Thus the number of subsets is the number of ways to select these elements.
2.

$$\begin{aligned}
\text{The number of subsets of } S = \;&(\text{the number of subsets of } S \text{ with 0 elements})\\
&+ (\text{the number of subsets of } S \text{ with 1 element})\\
&+ \cdots\\
&+ (\text{the number of subsets of } S \text{ with } n \text{ elements})\\
= \;&\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k}\\
= \;&(1+1)^n = 2^n.
\end{aligned}$$

□

Arrangements Around a Circle

Example 5. *Ten people are at a meeting around a circular table. How many different ways can they be seated?*

The main difficulty in this problem is that some arrangements are equivalent to others. Given some seating arrangement, we get an equivalent one if everyone just moves one seat over. Therefore, we cannot directly apply the ${}_nP_r$ formula.

One way around this difficulty is to not allow this shifting to occur. We can do this by fixing one of the people in some chair. That is, we seat one of the people and count the arrangements of the remaining nine relative to him. In this case, we are simply arranging nine people in nine chairs (and none of the arrangements are equivalent now) so we have ${}_9P_9 = 9!$ ways to do it.

Another way around this difficulty is to analyze the overcount if we ignore equivalent arrangements. In this case, we are arranging ten people in ten chairs and there are ${}_{10}P_{10} = 10!$ ways to do it. Now, for each of these arrangements, we get an equivalent arrangement if and only if everyone shifts clockwise or counterclockwise around the table. Since there are ten chairs, you can shift 10 times before you get back to the original configuration, if you shift one chair at a time. Therefore, every arrangement is overcounted exactly 10 times. So we have $10!/10 = 9!$ distinct seating arrangements.