MATH 2113 / CSCI 2113
Assignment # 5
Due Friday, March 30, 2007

Each problem is worth 10 points, for a total of 80 points.

1. Prove the following combinatorially. (Hint: count edges in a graph)
   (a) For all integers k and n such that $1 \leq k \leq n$,
   $$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$
   (b) For $n \in \mathbb{N}$ and any set of positive integers $\{n_1, \ldots, n_k\}$ such that $n_1 + \cdots + n_k = n$, we have
   $$\sum_{i=1}^{k} \binom{n_i}{2} \leq \binom{n}{2}.$$
2. (a) Prove that the complement of a disconnected graph is connected.
   (b) Show that the converse is false. That is, draw a connected graph whose complement is also connected.
3. (a) Prove that a bipartite graph is regular only if its partite sets have the same number of vertices.
   (b) How many edges are in the complete bipartite graph $K_{m,n}$?
4. Two players play a game on a graph G by alternately picking distinct vertices. Player 1 picks any vertex, then every subsequent choice must be adjacent to the preceding choice (of the other player). Thus together the two players follow a path. (A player may not choose a vertex that's been chosen earlier in the game). The player who cannot make a move loses. Prove that if G has a perfect matching, then Player 2 has a winning strategy.

   Note it is also true that if G does *not* have a perfect matching, then Player 1 has a winning strategy. (You do not have to do this part)
5. Show that the travelling salesman problem for an arbitrary weighted graph on n vertices can be reduced to the travelling salesman problem for a complete weighted graph on n vertices.
6. Prove that every simple graph G has a cycle of length at least $\delta(G) + 1$ if $\delta(G) \geq 2$. (Hint: use the result of Problem 3a on midterm 2)
7. Let T be a tree with $n \geq 3$ vertices. Prove that $\alpha(T) \geq$ the number of leaves (terminal vertices) in T.
8. Show that the number of full binary trees with $n+1$ leaves, T_{n+1}, is the n-th Catalan number C_n, by finding a recurrence relation for T_{n+1}. (Here, we distinguish between left and right

children, so that $T_3 = 2$. So the two trees below are considered distinct by this count.)