MATH 2113 / CSCI 2113
Assignment # 3
Due Friday, March 2, 2007

Note: The problems in this problem set will be marked out of 10 points each, except problem 3, which will be worth 5 points, for a total of 75 points. However, this problem set will be weighted the same as the other sets. The "bonus" part of problem 6 will be due with assignment 4.

1. Determine whether the following functions are one-to-one, onto, both, or neither. Prove or give counterexamples.
   Note: $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.

   (a) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, f(x) = (x+1)^2$

   (b) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, f(x) = (x+1)^2$

   (c) f : bit-strings of length $n \to \{0, 1, \ldots, n\}$, $f(x) =$ sum of digits of x
   (Hint: your answer will depend on the value of n)

   (d) $f : \mathbb{N} \to \mathbb{Z}, f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (1-n)/2 & \text{if } n \text{ is odd} \end{cases}$

2. Let A_n be the set of bit-srings of length n such that any block of consecutive zeroes has length 3. For example $100011 \in A_6$ but $000000 \notin A_6$. Let B_n be the set of bit-strings of length n with no consecutive zeroes. Show that $|A_n| \leq |B_n|$ by finding a one-to-one function $f : A_n \to B_n$. Prove that your function is indeed one-to-one by showing that different strings in A_n map to different strings in B_n.

3. Do problem 25 of section 7.2 of the text by rephrasing it. (That is, reduce the problem to one you've already done)

4. Consider the message "READ THIS MESSAGE".

   (a) How many bits are required to store the message in 8-bit ASCII?

   (b) Encode the message using a Huffman Coding scheme. How many bits are used? (Ignore the size of the dictionary)

5. If everyone at a party knows at least one other person, prove that there are two guests who know the same number of people.

6. For this problem, assume that x is an irrational number.

   (a) Let a_k denote the fractional part of kx (That is, $a_k = kx - \lfloor kx \rfloor$). Prove that among the $n+1$ numbers $a_1, a_2, \ldots, a_{n+1}$ there is a pair that is within $1/n$ of each other.

   (b) Let $N \in \mathbb{N}$ be given. Prove that there is an integer q, $1 \leq q \leq N$, such that qx is within $1/N$ of an integer. Conclude that for any N, there exists integers p and q with $q \leq N$ such that
   $$\left| x - \frac{p}{q} \right| < \frac{1}{qN}.$$

(c) (Bonus:) Use the result in the previous part to conclude that for any N, there exists integers p and q with $q > N$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

This is called Dirichlet's Approximation Theorem.

7. Let $n \geq 11$. Prove that it is possible to pick forty-one different ordered triples from $\{1, 2, \ldots, n\}$ that all have the same sum. (Assume you may pick the same number more than once in any triple. For example, the triples $(2, 2, 2)$, $(1, 2, 3)$, and $(3, 1, 2)$ are all allowed and considered different triples.)

8. (a) Six friends are comparing scores after a test. Suppose the average of their scores is 77 and two people scored under 60. Prove that someone must have scored over 85.

   (b) If the lowest two scores among the friends are both 59 on this test, prove that at least two people scored above 80 points. (Hint: use contradiction, and assume the highest possible score is 100)