MATH 2113 / CSCI 2113
Assignment # 2
Due Monday, February 5, 2007

1. Another popular Chinese game is "Yu-Xia-Xie", or, literally translated, "Fish-Shrimp-Crab". The game is played with three dice. On each die, instead of the numbers one through six, there are six pictures (three of which are those in the name). You may bet on any of the pictures. The house rolls the three dice, and the payout is x-to-1, where x is the number of times the picture you bet on shows up. For example, if the dice show fish-fish-crab and you had bet \$1 on fish, then you get \$2, plus your original wager returned. If your picture doesn't show up, you lose your bet.

   Find the expected value of a \$1 bet on any of the pictures in this game (say, fish).

2. Suppose there are n lines in the plane, such that every line crosses every other line at exactly one point, and no three meet at a common point. Let a_n denote the number of regions this divides the plane into. ($a_1 = 2, a_2 = 4, a_3 = 7$) Find and solve a recurrence relation for a_n.

3. Find and solve a recurrence relation for the number of ways to tile a $2 \times n$ checkerboard with identical dominoes.

4. Use generating functions to evaluate the following.

   (a) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$

   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k k^2$

5. Prove that the number of derangements of n objects, d_n, satisfies the recurrence relation

$$d_n = nd_{n-1} + (-1)^n$$

   and with the initial condition $d_1 = 0$ prove the formula

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

6. Let $f(x)$ be the generating function for $\{d_n\}$. Find a differential equation satisfied by $f(x)$

7. Find the partial fraction decomposition of the following functions, and give an explicit formula for the coefficient of x^n in their power series expansions.

   (a) $\displaystyle\frac{1}{(1-x)^2(1-3x)^2}$

   (b) $\displaystyle\frac{1}{(1-x)(1+x^2)}$

8. Find the generating function $f(x)$ for the sequence $\{a_n\}_{n=0}^{\infty}$ given by

$$a_n = 2a_{n-1} + 3a_{n-2} + 4n - 1,$$

and $a_0 = 1$, $a_1 = 0$.

9. Recall the spinner problem from Assignment 1 (Problem 5). Consider a spinner with n regions labelled $1, 2, \ldots, n$, spun three times. Let a_k denote the number of ways to obtain a sum of k (e.g. if $k = 4$, the spins 1-2-1 and 2-1-1 are distinct). Find the generating function for $\{a_k\}$.

10. Suppose you want to buy cookies, and there are three types to choose from. They cost \$2, \$3, and \$5. Let a_n be the number of ways you can spend exactly \$$n$ buying cookies.

    (a) Find the generating function for $\{a_n\}$.

    (b) Suppose the store only has ten \$2 cookies left, let b_n be the number ways to spend exactly \$$n$ in this case. Find the generating function for $\{b_n\}$